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THE FINITE LENGTH PARTIAL FITTED JOURNAL BEARING WITH AXIAL OIL GROOVE*

BY

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1. Introduction. Reynolds' lubrication theory gives rise to a differential equation for the pressure distribution in a lubricating film. This film is usually restricted to lie between a stationary surface and a moving surface. In any practical problem, the two surfaces are, of course, of finite dimensions. However, all early solutions of Reynolds' equation involved the assumption of one infinite dimension since this was equivalent to not permitting fluid flow in this direction, thereby reducing the problem to the solution of an ordinary differential equation. The results have severe limitations, however. In the case of journal bearings, the reduction of the axial length from infinity to a finite value can be expected to decrease the circumferential extent of the lubricating film, decrease the pressure developed throughout the film, and cause the point of closest approach of the journal and bearing surfaces to move closer to the plane of the applied load. These changes will in turn affect the performance characteristics of the journal bearing. For example, Shaw and Macks [1]** have compared properties of the full journal bearing predicted by the one-dimensional theory with measured values found by Brad-

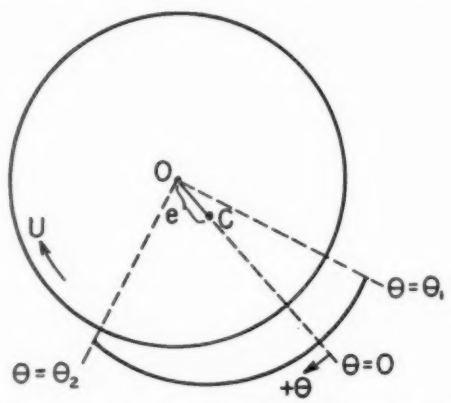

FIG. 1. Cross-section of a partial fitted journal bearing.

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**Numbers in square brackets refer to the bibliography at the end of the paper.

ford [2] and have concluded that the simple theory can lead to optimistic values of load capacity that err by more than 100 per cent.

It is the purpose of this paper to determine an analytical expression for the pressure distribution in the lubricating film of a finite length partial fitted journal bearing when the lubricant is introduced under constant pressure through an axial bearing groove. A partial bearing is one in which the bearing does not completely surround the journal, and the term "fitted" implies that the radii of the journal and bearing are equal. A cross-section of such a journal bearing is shown in Fig. 1 where O and C are the centers of the journal and bearing, respectively, e is the eccentricity, U is the linear velocity of the journal, and θ is an angle measured from the directed line of centers OC. The angle θ_1 is called the inlet edge of the bearing surface and θ_2 is the outlet edge.

2. Reynolds' equation and its solution. For an incompressible and constant viscosity lubricant, Reynolds' equation for a finite length journal bearing may be written as [3]

$$\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(h^3\frac{\partial p}{\partial\theta}\right)+\frac{\partial}{\partial y}\left(h^3\frac{\partial p}{\partial y}\right)=\frac{6\mu U}{r}\frac{dh}{d\theta}, \tag{2.1}$$

where p, μ, and h are the pressure, viscosity, and radial thickness of the lubricant, y is the axial variable measured from the central axial section of the journal bearing, and r is the common radius of the journal and bearing. If h is replaced by $e\cos\theta$ and if the dimensionless variable $w = y/r$ is introduced, then (2.1) reduces to

$$\frac{\partial}{\partial\theta}\left(\cos^3\theta\frac{\partial p}{\partial\theta}\right)+\frac{\partial}{\partial w}\left(\cos^3\theta\frac{\partial p}{\partial w}\right)=\beta\frac{d\cos\theta}{d\theta}, \tag{2.2}$$

where $\beta = 6\mu Ur/e^2$.

In order to remove the inhomogeneous term in (2.2) we set

$$p(\theta, w) = P(\theta, w) + f(\theta). \tag{2.3}$$

Substituting this expression into (2.2) and separating terms in θ alone from those in θ and w gives rise to two differential equations. The function $f(\theta)$ satisfies

$$\frac{d}{d\theta}\left(\cos^3\theta\frac{df}{d\theta}\right)=\beta\frac{d\cos\theta}{d\theta} \tag{2.4}$$

and $P(\theta, w)$ satisfies

$$\frac{\partial}{\partial\theta}\left(\cos^3\theta\frac{\partial P}{\partial\theta}\right)+\frac{\partial}{\partial w}\left(\cos^3\theta\frac{\partial P}{\partial w}\right)=0. \tag{2.5}$$

Equation (2.4) is simply the differential equation for the pressure distribution in the corresponding infinite length bearing and its solution is readily found to be

$$\begin{aligned} f(\theta) &= \beta\int_0^\theta\frac{d\theta}{\cos^2\theta}+K_1\int_0^\theta\frac{d\theta}{\cos^3\theta}+K_2 \\ &= \beta g(\theta)+K_1k(\theta)+K_2, \end{aligned} \tag{2.6}$$

where

$$g(\theta) = \tan\theta, \tag{2.7}$$

$$k(\theta) = \frac{\tan\theta}{2\cos\theta}+\frac{1}{4}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right) \tag{2.8}$$

and K_1 and K_2 are constants of integration. The solution to (2.5) represents a term in the pressure solution which corrects for finiteness of the bearing or lubricant flow through the end axial sections. It may be determined by seeking a solution having the form

$$P(\theta, w) = Q(\theta)W(w). \tag{2.9}$$

If (2.9) is substituted into (2.5) and the variables are separated, it is found that W and Q satisfy the equations

$$W'' - \lambda^2 W = 0 \tag{2.10}$$

and

$$Q'' - 3 \tan \theta Q' + \lambda^2 Q = 0, \tag{2.11}$$

where $-\lambda^2$ was chosen to be the separation constant. Equation (2.11) together with the boundary conditions to be applied here is a problem of the Sturm-Liouville type and gives rise to a sequence of solutions Q_i corresponding to a sequence of real, non-negative eigenvalues λ_i^2. We therefore write the solution of (2.10) as

$$W_i = A_i \cosh (\lambda_i w) + B_i \sinh (\lambda_i w) \tag{2.12}$$

for $i \neq 0$ ($\lambda_i \neq 0$) and

$$W_0 = A_0 + B_0 w \tag{2.13}$$

for $i = 0$ ($\lambda_i = \lambda_0 = 0$). In order to solve (2.11) we make the substitution $z = \sin^2\theta$ which results in the equation

$$z(1 - z)Q_i'' + \left(\frac{1}{2} - \frac{5}{2}z\right)Q_i' + \frac{\lambda_i^2}{4} Q_i = 0. \tag{2.14}$$

This is the hypergeometric differential equation and has as its solution, for $i \neq 0$,

$$\begin{aligned} Q_i(\theta) &= C_i F(a_i, b_i; \tfrac{1}{2}; \sin^2 \theta) + D_i \sin \theta F(a_i + \tfrac{1}{2}, b_i + \tfrac{1}{2}; \tfrac{3}{2}; \sin^2 \theta) \\ &= C_i F_i^{(1)}(\theta) + D_i \sin \theta F_i^{(2)}(\theta), \end{aligned} \tag{2.15}$$

where

$$a_i = \tfrac{1}{4}[3 + (9 + 4\lambda_i^2)^{1/2}] \tag{2.16}$$

and

$$b_i = \tfrac{1}{4}[3 - (9 + 4\lambda_i^2)^{1/2}]. \tag{2.17}$$

The above result is valid for $z = \sin^2\theta < 1$ and this is consistent with the physical requirement that θ must lie in the open interval $-\pi/2, \pi/2$ for there to be a non-zero film thickness at all points between the journal and the bearing. When $i = 0$ ($\lambda_i = \lambda_0 = 0$) the corresponding solution Q_0 is given by

$$\begin{aligned} Q_0 &= C_0 \int_0^\theta \frac{d\theta}{\cos^3 \theta} + D_0 \\ &= C_0\left[\frac{\tan \theta}{2 \cos \theta} + \frac{1}{4} \log \left(\frac{1 + \sin \theta}{1 - \sin \theta}\right)\right] + D_0 . \end{aligned} \tag{2.18}$$

The general solution of (2.5) may therefore be written as

$$P(\theta, w) = Q_0 W_0 + \sum_{i=1}^{\infty} W_i(w) Q_i(\theta). \tag{2.19}$$

3. Application of boundary conditions. The simplest example of a journal bearing having an axial oil inlet is that where the lubricant is introduced at constant pressure p_0 at the inlet edge $\theta = \theta_1$. For this problem, the boundary conditions to be satisfied by the pressure are

$$\begin{aligned} p(\theta_1, w) &= p_0, \\ p(\theta_2, w) &= 0, \\ p(\theta, w_0) &= p(\theta, -w_0) = 0, \end{aligned} \tag{3.1}$$

where $w_0 = L/r$ and $2L$ is the total length of the bearing. In terms of $f(\theta)$ and $P(\theta, w)$ we require that

$$\begin{aligned} f(\theta_1) &= p_0 \\ f(\theta_2) &= 0 \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} P(\theta_1, w) &= P(\theta_2, w) = 0 \\ P(\theta, w_0) &= P(\theta, -w_0) = -f(\theta). \end{aligned} \tag{3.3}$$

With the use of these boundary conditions, we can determine the coefficients K_1, K_2, A_i, B_i, C_i, and D_i as well as the eigenvalues λ_i^2. From (2.6) and (3.2) it is easily shown that

$$K_1 = -\frac{\beta[g(\theta_2) - g(\theta_1)] + p_0}{k(\theta_2) - k(\theta_1)} \tag{3.4}$$

and

$$K_2 = -\frac{\beta[g(\theta_1)k(\theta_2) - g(\theta_2)k(\theta_1)] - p_0 k(\theta_2)}{k(\theta_2) - k(\theta_1)}. \tag{3.5}$$

In order to satisfy the first two conditions of (3.3) we require that each $Q_i(\theta)$ satisfy

$$Q_i(\theta_1) = Q_i(\theta_2) = 0. \tag{3.6}$$

From (2.15) and (3.6) we have the two equations

$$C_i F_i^{(1)}(\theta_1) + D_i \sin \theta_1 F_i^{(2)}(\theta_1) = 0 \tag{3.7}$$

and

$$C_i F_i^{(1)}(\theta_2) + D_i \sin \theta_2 F_i^{(2)}(\theta_2) = 0 \tag{3.8}$$

which may be combined to give

$$C_i[\sin \theta_1 F_i^{(1)}(\theta_2) F_i^{(2)}(\theta_1) - \sin \theta_2 F_i^{(1)}(\theta_1) F_i^{(2)}(\theta_2)] = 0 \tag{3.9}$$

and

$$D_i[\sin \theta_1 F_i^{(1)}(\theta_2) F_i^{(2)}(\theta_1) - \sin \theta_2 F_i^{(1)}(\theta_1) F_i^{(2)}(\theta_2)] = 0. \tag{3.10}$$

Since C_i and D_i cannot both be zero for a non-trivial solution we require that

$$\sin \theta_1 F_i^{(1)}(\theta_2) F_i^{(2)}(\theta_1) - \sin \theta_2 F_i^{(1)}(\theta_1) F_i^{(2)}(\theta_2) = 0. \tag{3.11}$$

This is an equation for the eigenvalues λ_i^2. The use of this result is reserved for Sec. 4. With the λ_i^2 determined, it is now possible to reduce the two unknown constants C_i and D_i in (2.15) to only one unknown by making use of (3.7) or (3.8). In order to determine C_0 and D_0 we apply (3.6) to (2.18) to get the two equations

$$C_0 \int_0^{\theta_1} \frac{d\theta}{\cos^3 \theta} + D_0 = 0 \tag{3.12}$$

and

$$C_0 \int_0^{\theta_1} \frac{d\theta}{\cos^3 \theta} + C_0 \int_{\theta_1}^{\theta_2} \frac{d\theta}{\cos^3 \theta} + D_0 = 0. \tag{3.13}$$

Subtracting these expressions gives $C_0 = 0$ after which $D_0 = 0$ from (3.12). Consequently, there is no solution corresponding to $i = 0$.

The remaining coefficients may now be evaluated by making use of the last two conditions of (3.3). From (2.12), (2.19), and (3.3) we have the two equations

$$\sum_{i=1}^{\infty} [A_i \cosh (\lambda_i w_0) + B_i \sinh (\lambda_i w_0)] Q_i(\theta) = -f(\theta) \tag{3.14}$$

and

$$\sum_{i=1}^{\infty} [A_i \cosh (\lambda_i w_0) - B_i \sinh (\lambda_i w_0)] Q_i(\theta) = -f(\theta), \tag{3.15}$$

where it is to be understood that the single unknown coefficient in $Q_i(\theta)$ has been incorporated into the A_i and B_i. Before employing these expressions, we first note from (2.11) and (3.6) that the $Q_i(\theta)$ are orthogonal with respect to the weighing function $\cos^3\theta$ over the interval θ_1, θ_2. That is,

$$\int_{\theta_1}^{\theta_2} \cos^3 \theta Q_i(\theta) Q_j(\theta)\, d\theta = 0 \tag{3.16}$$

for $i \neq j$. If (3.14) and (3.15) are multiplied by $\cos^3\theta\, Q_j(\theta)$ and the results integrated over θ_1, θ_2 two equations in A_i and B_i result which, when solved simultaneously, give

$$A_i = -\frac{\displaystyle\int_{\theta_1}^{\theta_2} f(\theta) \cos^3 \theta Q_i(\theta)\, d\theta}{\cosh (\lambda_i w_0) \displaystyle\int_{\theta_1}^{\theta_2} \cos^3 \theta Q_i^2(\theta)\, d\theta} \tag{3.17}$$

and

$$B_i = 0. \tag{3.18}$$

The latter result makes the pressure solution an even function in w which is to be expected from the symmetry of the problem. This completes the evaluation of the coefficients, and the pressure solution

$$p(\theta, w) = f(\theta) + \sum_{i=1}^{\infty} W_i(w) Q_i(\theta) \tag{3.19}$$

is now completely determined for the case where the lubricant enters at constant pressure through the inlet edge of the journal bearing.

For the case where the lubricant enters through an axial bearing groove at θ_G where $\theta_1 < \theta_G < \theta_2$, the analysis is analogous to that preceding except that two pressure solutions of the type (3.19) must be developed, one for the range θ_1, θ_G and another for the range θ_G, θ_2. Once the pressure distribution has been determined, it is possible to compute such quantities as load carrying capacity, coefficient of friction, axial thrust, and rate of flow of lubricant. These will not be discussed here, however, since their definitions are readily available in any one of numerous lubrication texts.

4. Evaluation of eigenvalues. In the preceding development, (3.11) gives the eigenvalues λ_i^2. In this section one technique for determining them will be described which employs Whittaker's method for finding the root of smallest absolute value of a power series [4].

We first recall that the hypergeometric function may be expanded in the form

$$F(a, b; c; x) = 1 + \frac{a}{1}\frac{b}{c}x + \frac{a}{1}\frac{b}{c}\frac{a+1}{2}\frac{b+1}{c+1}x^2 + \cdots \tag{4.1}$$

The functions $F_i^{(1)}(\theta)$ and $F_i^{(2)}(\theta)$ are easily expressed in this form if we note that the coefficients in (4.1) contain terms of the type

$$(a + n)(b + n) = ab + n(a + b) + n^2, \tag{4.2}$$

where n is zero or any positive integer. From (2.16) and (2.17) we have that $ab = -\lambda_i^2/4$ and $a + b = 3/2$ so that, for $F_i^{(1)}(\theta)$,

$$(a + n)(b + n) = -\frac{\lambda_i^2}{4} + \frac{n}{2}(2n + 3). \tag{4.3}$$

Similarly, for $F_i^{(2)}(\theta)$,

$$(a + n)(b + n) = -\frac{\lambda_i^2}{4} + \frac{1}{2}(n + 2)(2n + 1). \tag{4.4}$$

The coefficients in the expansions of $F_i^{(1)}(\theta)$ and $F_i^{(2)}(\theta)$ are therefore polynomials in λ_i^2. These expansions may now be substituted into (3.11) and the series rearranged to give a power series in λ_i^2 where each coefficient is, in general, a rapidly converging infinite series. If we represent the result as

$$e_0 + e_1\lambda_i^2 + e_2(\lambda_i^2)^2 + e_3(\lambda_i^2)^3 + \cdots = 0, \tag{4.5}$$

then, by Whittaker's method, the smallest root λ_1^2 is given by

$$-\frac{e_0}{e_1} - \frac{e_0^2 e_2}{e_1}\begin{vmatrix} e_1 & e_2 \\ e_0 & e_1 \end{vmatrix}^{-1} - e_0^3\begin{vmatrix} e_2 & e_3 \\ e_1 & e_2 \end{vmatrix}\begin{vmatrix} e_1 & e_2 \\ e_0 & e_1 \end{vmatrix}^{-1}\begin{vmatrix} e_1 & e_2 & e_3 \\ e_0 & e_1 & e_2 \\ 0 & e_0 & e_1 \end{vmatrix}^{-1} - \cdots \tag{4.6}$$

The second root may now be found by factoring out the first root and repeating the procedure, and similarly for all other roots.

5. Asymptotic forms. Because of the complex nature of the eigenfunctions and Eq. (3.11) for the eigenvalues, it is convenient to develop approximate forms which

are valid for large values of λ_i. Watson has derived asymptotic expansions for $F(a, b; c; x)$ when various combinations of the parameters are large [5]. When applied to $F_i^{(1)}(\theta)$, the appropriate expansion gives

$$F_i^{(1)}(\theta) \sim \frac{2 \cos [\frac{1}{2}\theta(9 + 4\lambda_i^2)^{1/2}]}{(9 + 4\lambda_i^2)^{1/2} \cos^{3/2} \theta} \frac{\Gamma(\frac{1}{4}[1 + \{9 + 4\lambda_i^2\}^{1/2}])}{\Gamma(\frac{1}{4}[-1 + \{9 + 4\lambda_i^2\}^{1/2}])}, \tag{5.1}$$

where Γ represents the gamma function. If all functions in λ_i are expanded and only terms of order 1 are kept, (5.1) reduces to

$$F_i^{(1)}(\theta) \sim \frac{\cos \lambda_i \theta}{\cos^{3/2} \theta}. \tag{5.2}$$

Similarly, it can be shown that

$$F_i^{(2)}(\theta) \sim \frac{\sin \lambda_i \theta}{\lambda_i \sin \theta \cos^{3/2} \theta}. \tag{5.3}$$

If these results are substituted into (3.11), it is found that

$$\lambda_i \sim \frac{i\pi}{\theta_2 - \theta_1}. \tag{5.4}$$

The same result can, of course, be determined from the general theory of the asymptotic behavior of the eigenvalues in the Sturm-Liouville problem [6].

6. **Example.** In order to give some indication of the nature of the solution determined above, we shall consider the simple case where the lubricant enters under constant pressure at $\theta = \theta_1$ and we shall suppose that $\theta_1 = -\theta_2$. That is, the clearance between the journal and the bearing surface will be taken to be symmetric about the line of centers $\theta = 0$. In addition, we shall take θ_2 to be small. This corresponds to a small bearing arc and makes λ_i large from (5.4), thereby justifying the use of the asymptotic forms developed above.

From (3.4) and (3.5) we have that

$$K_1 = -\frac{2\beta g(\theta_2) + p_0}{2k(\theta_2)} \tag{6.1}$$

and

$$K_2 = \frac{p_0}{2}, \tag{6.2}$$

so that, from (2.6),

$$f(\theta) = \beta g(\theta) - \frac{2\beta g(\theta_2) + p_0}{2k(\theta_2)} k(\theta) + \frac{p_0}{2}. \tag{6.3}$$

It will be recalled that this is not only a portion of the finite length bearing solution, but is also the pressure distribution in the corresponding infinite length bearing.

From (5.4) the square root of the first eigenfunction is given by

$$\lambda_1 \sim \frac{\pi}{2\theta_2} \tag{6.4}$$

and (2.15), (5.2), (5.3), and (6.4) give

$$Q_1(\theta) \sim C_1 \frac{\cos\left[\pi\theta/(2\theta_2)\right]}{\cos^{3/2}\theta} + D_2 \frac{2\theta_2 \sin\left[\pi\theta/(2\theta_2)\right]}{\pi \cos^{3/2}\theta}. \tag{6.5}$$

From (3.6) it is found that $D_1 = 0$ so that

$$Q_1(\theta) \sim C_1 \frac{\cos\left[\pi\theta/(2\theta_2)\right]}{\cos^{3/2}\theta}. \tag{6.6}$$

Similarly,

$$Q_2(\theta) \sim D_1 \frac{\theta_2 \sin\left(\pi\theta/\theta_2\right)}{\pi \cos^{3/2}\theta} \tag{6.7}$$

and the pressure solution from (3.19) is

$$p(\theta, w) \sim \beta g(\theta) - \frac{2\beta g(\theta_2) + p_0}{2k(\theta_2)} k(\theta) + \frac{p_0}{2} + A_1 \cosh\left(\frac{\pi}{2\theta_2} w\right) \frac{\cos\left[\pi\theta/2\theta_2\right]}{\cos^{3/2}\theta} + A_2 \frac{\theta_2}{\pi} \cosh\left(\frac{\pi}{\theta_2} w\right) \frac{\sin\left(\pi\theta/\theta_2\right)}{\cos^{3/2}\theta} + \cdots, \tag{6.8}$$

where $g(\theta)$ and $h(\theta)$ are given by (2.7) and (2.8), and, from (3.17),

$$A_1 = \frac{-p_0}{\theta_2 \cosh\left(\frac{\pi}{2\theta_2} w_0\right)} \int_0^{\theta_2} \cos^{3/2}\theta \cos\frac{\pi}{2}\frac{\theta}{\theta_2}\, d\theta \tag{6.9}$$

and

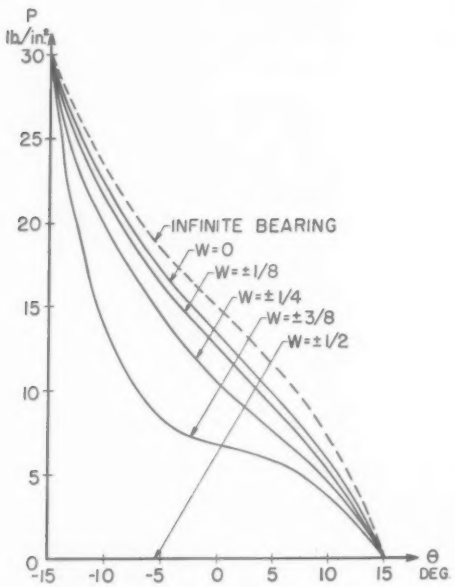

FIG. 2. Circumferential variation of pressure.

$$A_2 = \frac{-2\pi}{\theta_2^2 \cosh(\pi w_0/\theta_2)} \int_0^{\theta_2} [\beta g(\theta) + K_1 k(\theta)] \cos^{3/2}\theta \sin \pi \frac{\theta}{\theta_2}\, d\theta. \tag{6.10}$$

The integrals in these expressions are easily evaluated by successive integrations by parts if we recall that $\pi/2\theta_2$ was assumed large.

We shall now apply (6.8) to the specific problem where

radius of bearing $= r = 1$ in.
length of bearing $= 2L = 1$ in.
speed $= 2000$ r.p.m.
eccentricity $= e = .001$ in.
viscosity $= \mu = 1.2 \times 10^{-6}$ reyn
oil inlet pressure $= p_0 = 30$ lb./in.^{2}
outlet edge of bearing $= \theta_2 = 15°$

With these conditions, the pressure solution to two eigenfunctions reduces to

$$p(\theta, w) \sim 1510 \tan\theta - 387\left[2\frac{\tan\theta}{\cos\theta} + \log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)\right] + 15 - 1.80 \cosh(6w)\frac{\cos 6\theta}{\cos^{3/2}\theta} + .0452 \cosh(12w)\frac{\sin 12\theta}{\cos^{3/2}\theta}. \tag{6.11}$$

This result can be expected to represent a good approximation to the complete pressure solution for all w not close to $w_0 = \pm \frac{1}{2}$ since, in these neighborhoods, the hyperbolic functions in (6.11) are relatively large. However, in these cases, the behavior of the pressure solution is known from the boundary conditions.

Equation (6.11) appears graphically in Figs. 2 and 3. Figure 2 illustrates how the

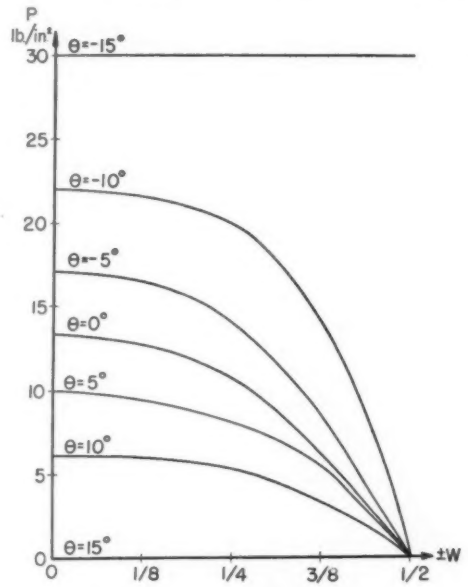

FIG. 3. Axial variation of pressure.

pressure distribution at various axial sections differs from that for the corresponding infinite bearing. While the distribution at the central axial section $w = 0$ does not differ greatly from the infinite distribution in this case, the difference would be considerable had the bearing length been taken shorter or the bearing arc larger. Figure 3 shows the variation in pressure as a function of axial distance for various fixed values of θ. The pressure decreases from a peak value at the central axial section to zero at both end sections. The corresponding pressure for the infinite bearing would be, of course, constant for all w and each fixed θ.

Bibliography

1. M. C. Shaw and E. F. Macks, *Analysis and lubrication of bearings*, McGraw-Hill, New York, 1949
2. L. J. Bradford, *Bearing design in the light of oil film pressure investigations*, Penna. State Coll. Bull. 14, 22 (1931)
3. A. E. Norton, *Lubrication*, McGraw-Hill, New York, 1942
4. E. T. Whittaker, *A formula for the solution of algebraic or transcendental equations*, Proc. Edinburgh Math. Soc. **36**, 103 (1917-18)
5. G. N. Watson, *Asymptotic expansions of hypergeometric functions*, Trans. Cambridge Phil. Soc. **22**, 277-308 (1918)
6. R. Courant and D. Hilbert, *Methods of mathematical physics*, Interscience Publishers, New York, 1953

A CLASS OF PROBLEMS ON LONGITUDINAL VIBRATIONS*

BY

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1. Introduction. Consider a bar of length l and constant area q formed of n parts with lengths l_k, densities ρ_k and Young's moduli E_k ($k = 1, 2, \cdots, n$). Denoting by $u_k(x, t)$ the displacement in longitudinal vibrations at the point x of the kth part and at any time t, the general form of the Laplace transform

$$U_k(x, p) = p \int_0^\infty e^{-pt} u_k(x, t)\, dt; \qquad k = 1, 2, \cdots, n$$

of $u_k(x, t)$ is

$$U_k(x, p) = A_k \cosh \frac{px}{a_k} + B_k \sinh \frac{px}{a_k} + W_k(x, p), \quad a_k = \sqrt{\frac{E_k}{\rho_k}}; \quad k = 1, 2, \cdots, n. \tag{1}$$

There exists a vast class of cases, important from the technical and physical points of view, where

$$W_k(s_k, p) = W_{k+1}(s_k, p), \qquad W_k'(s_k, p) = c_k W_{k+1}'(s_k, p);$$

$$s_k = \sum_{\kappa=1}^{k} l_\kappa, \qquad c_k = \frac{E_{k+1}}{E_k}, \qquad W_k' = \frac{dW_k}{dx}; \qquad k = 1, 2, \cdots, n-1. \tag{2}$$

All problems of this group may be treated in the following manner.

2. General theory. The subsidiary equations corresponding to the well-known conditions at the sections separating adjacent parts of the rod are (primes denote the x-derivatives):

$$U_k(s_k, p) = U_{k+1}(s_k, p), \qquad U_k'(s_k, p) = c_k U_{k+1}'(s_k, p); \qquad k = 1, 2, \cdots, n-1. \tag{3}$$

Substituting from (1) and using (2) we have

$$A_k \cosh \frac{ps_k}{a_k} + B_k \sinh \frac{ps_k}{a_k} = A_{k+1} \cosh \frac{ps_k}{a_{k+1}} + B_{k+1} \sinh \frac{ps_k}{a_{k+1}}$$

$$A_k \sinh \frac{ps_k}{a_k} + B_k \cosh \frac{ps_k}{a_k} = h_k\left(A_{k+1} \sinh \frac{ps_k}{a_{k+1}} + B_{k+1} \cosh \frac{ps_k}{a_{k+1}}\right), \tag{3.1}$$

$$h_k = \frac{a_{k+1}\rho_{k+1}}{a_k \rho_k}; \qquad k = 1, 2, \cdots, n-1,$$

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or, in the matrix form,

$$\left\| \begin{matrix} A_{k+1} \\ B_{k+1} \end{matrix} \right\| = M_k(p) \left\| \begin{matrix} A_k \\ B_k \end{matrix} \right\|; \qquad k = 1, 2, \cdots, n-1$$

$$M_k(p) = \left\| \begin{matrix} \cosh \frac{ps_k}{a_k} \cosh \frac{ps_k}{a_{k+1}} - \frac{1}{h_k} \sinh \frac{ps_k}{a_k} \sinh \frac{ps_k}{a_{k+1}}, & \sinh \frac{ps_k}{a_k} \cosh \frac{ps_k}{a_{k+1}} - \frac{1}{h_k} \cosh \frac{ps_k}{a_k} \sinh \frac{ps_k}{a_{k+1}} \\ -\cosh \frac{ps_k}{a_k} \sinh \frac{ps_k}{a_{k+1}} + \frac{1}{h_k} \sinh \frac{ps_k}{a_k} \cosh \frac{ps_k}{a_{k+1}}, & -\sinh \frac{ps_k}{a_k} \sinh \frac{ps_k}{a_{k+1}} + \frac{1}{h_k} \cosh \frac{ps_k}{a_k} \cosh \frac{ps_k}{a_{k+1}} \end{matrix} \right\|;$$

$$k = 1, 2, \cdots, n-1. \qquad (4)$$

It follows from this that

$$\left\| \begin{matrix} A_{k+1} \\ B_{k+1} \end{matrix} \right\| = Q_k(p) \left\| \begin{matrix} A_1 \\ B_1 \end{matrix} \right\|; \qquad k = 1, 2, \cdots, n-1,$$

$$Q_k(p) = M_k(p) M_{k-1}(p) \cdots M_2(p) M_1(p); \qquad k = 1, 2, \cdots, n-1. \qquad (4.1)$$

Writing

$$Q_k(p) = \| \, q_k^{(\mu\nu)}(p) \, \|, \qquad \mu, \nu = 1, 2; \qquad k = 1, 2, \cdots, n-1 \qquad (5)$$

we obtain from (4.1)

$$A_{k+1} = q_k^{(11)}(p) A_1 + q_k^{(12)}(p) B_1, \qquad B_{k+1} = q_k^{(21)}(p) A_1 + q_k^{(22)}(p) B_1;$$

$$k = 1, 2, \cdots, n-1$$

and the expressions (1) become

$$U_1(x, p) = A_1 \cosh \frac{px}{a_1} + B_1 \sinh \frac{px}{a_1} + W_1(x, p) \qquad (6.1)$$

$$U_{k+1}(x, p) = \left[q_k^{(11)}(p) \cosh \frac{px}{a_{k+1}} + q_k^{(21)}(p) \sinh \frac{px}{a_{k+1}} \right] A_1 + \left[q_k^{(12)}(p) \cosh \frac{px}{a_{k+1}} + q_k^{(22)}(p) \sinh \frac{px}{a_{k+1}} \right] B_1 + W_{k+1}(x, p);$$

$$k = 1, 2, \cdots, n-1. \qquad (6.2)$$

Transforming the conditions at the ends of the bar leads to the equations

$$\begin{aligned} \alpha_0 U_1(0, p) + \beta_0 U_1'(0, p) &= \gamma_0, \\ \alpha_n U_n(l, p) + \beta_n U_n'(l, p) &= \gamma_n, \end{aligned} \qquad (7)$$

where $\alpha_0, \beta_0, \cdots, \beta_n, \gamma_n$ depend usually on p.

By means of (6.1), (6.2) relations (7) yield a system of two equations for the un-

knowns A_1 , B_1 . Solving it and substituting the resultant values A_1 , B_1 into (6.1) and (6.2) leads to the following complicated expressions:

$$U_1(x, p) = \frac{1}{\Delta}\left(\Delta_1 \cosh\frac{px}{a_1} + \Delta_2 \sinh\frac{px}{a_1}\right) + W_1(x, p) \tag{8.1}$$

$$U_{k+1}(x, p) = \frac{1}{\Delta}\left\{\Delta_1\left[q_k^{(11)}(p) \cosh\frac{px}{a_{k+1}} + q_k^{(21)}(p) \sinh\frac{px}{a_{k+1}}\right]\right.$$
$$\left. + \Delta_2\left[q_k^{(12)}(p) \cosh\frac{px}{a_{k+1}} + q_k^{(22)}(p) \sinh\frac{px}{a_{k+1}}\right]\right\} + W_{k+1}(x, p);$$
$$k = 1, 2, \cdots, n-1 \tag{8.2}$$

$$\Delta_1 = \left[\left(\alpha_n \cosh\frac{pl}{a_n} + \frac{p}{a_n}\beta_n \sinh\frac{pl}{a_n}\right)q_{n-1}^{(12)}(p) + \left(\alpha_n \sinh\frac{pl}{a_n}\right.\right.$$
$$\left.\left. + \frac{p}{a_n}\beta_n \cosh\frac{pl}{a_n}\right)q_{n-1}^{(22)}(p)\right][\gamma_0 - \alpha_0 W_1(0, p) - \beta_0 W_1'(0, p)]$$
$$- \frac{p}{a_1}\beta_0[\gamma_n - \alpha_n W_n(l, p) - \beta_n W_n'(l, p)]$$

$$\Delta_2 = \alpha_0[\gamma_n - \alpha_n W_n(l, p) - \beta_n W_n'(l, p)] - \left[\left(\alpha_n \cosh\frac{pl}{a_n}\right.\right.$$
$$\left.\left. + \frac{p}{a_n}\beta_n \sinh\frac{pl}{a_n}\right)q_{n-1}^{(11)}(p) + \left(\alpha_n \sinh\frac{pl}{a_n}\right.\right.$$
$$\left.\left. + \frac{p}{a_n}\beta_n \cosh\frac{pl}{a_n}\right)q_{n-1}^{(21)}(p)\right][\gamma_0 - \alpha_0 W_1(0, p) - \beta_0 W_1'(0, p)]$$

$$\Delta = \left[\alpha_0 q_{n-1}^{(12)}(p) - \frac{p}{a_1}\beta_0 q_{n-1}^{(11)}(p)\right]\left(\alpha_n \cosh\frac{pl}{a_n} + \frac{p}{a_n}\beta_n \sinh\frac{pl}{a_n}\right)$$
$$+ \left[\alpha_0 q_{n-1}^{(22)}(p) - \frac{p}{a_1}\beta_0 q_{n-1}^{(21)}(p)\right]\left(\alpha_n \sinh\frac{pl}{a_n} + \frac{p}{a_n}\beta_n \cosh\frac{pl}{a_n}\right). \tag{8.3}$$

The formulae (8.1) to (8.3) determine the Laplace transforms of the functions $u_k(x, t)$ which are to be found by the methods of operational calculus. The most difficult part of the problem lies in finding an explicit form of the matrix (4.1).

3. An interesting class of problems. We now proceed to treating problems characterised by the equations $h_k = 1$ ($k = 1, 2, \cdots, n-1$), or, by (1) and (3.1),

$$E_k\rho_k = \text{const}; \qquad k = 1, 2, \cdots, n. \tag{9}$$

In this case it is possible to calculate the matrix (4.1) and our former results (8.1) to (8.3) become simpler. Physically the simplification arises from the fact that there is no reflection at the joints of the different parts.

Using the notation

$$\sigma_k = s_k\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right), \qquad \tau_k = \sum_{\kappa=1}^{k}\sigma_\kappa\,; \qquad k = 1, 2, \cdots, n-1 \tag{10}$$

we easily obtain from (4) and (4.1)

$$M_k(p) = \begin{Vmatrix} \cosh p\sigma_k\,, & \sinh p\sigma_k \\ \sinh p\sigma_k\,, & \cosh p\sigma_k \end{Vmatrix}; \qquad k = 1, 2, \cdots, n-1$$
$$Q_2(p) = \begin{Vmatrix} \cosh p\tau_2\,, & \sinh p\tau_2 \\ \sinh p\tau_2\,, & \cosh p\tau_2 \end{Vmatrix}. \tag{11}$$

However,

$$M_{\nu+1}(p) \begin{Vmatrix} \cosh p\tau_\nu\,, & \sinh p\tau_\nu \\ \sinh p\tau_\nu\,, & \cosh p\tau_\nu \end{Vmatrix} = \begin{Vmatrix} \cosh p\tau_{\nu+1}\,, & \sinh p\tau_{\nu+1} \\ \sinh p\tau_{\nu+1}\,, & \cosh p\tau_{\nu+1} \end{Vmatrix},$$

so that (4.1) yields the closed forms

$$Q_k(p) = \begin{Vmatrix} \cosh p\tau_k\,, & \sinh p\tau_k \\ \sinh p\tau_k\,, & \cosh p\tau_k \end{Vmatrix}; \qquad k = 1, 2, \cdots, n-1 \tag{12}$$

of the fundamental matrices $Q_k(p)$.

Equations (8.3) then give

$$\Delta_1 = \left[\alpha_n \sinh p\left(\frac{l}{a_n} + \tau_{n-1}\right) + \frac{p}{a_n}\beta_n \cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right)\right][\gamma_0 - \alpha_0 W_1(0, p) - \beta_0 W_1'(0, p)] - \frac{p}{a_1}\beta_0[\gamma_n - \alpha_n W_n(l, p) - \beta_n W_n'(l, p)]$$

$$\Delta_2 = \alpha_0[\gamma_n - \alpha_n W_n(l, p) - \beta_n W_n'(l, p)] - \left[\alpha_n \cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right) + \frac{p}{a_n}\beta_n \sinh p\left(\frac{l}{a_n} + \tau_{n-1}\right)\right][\gamma_0 - \alpha_0 W_1(0, p) - \beta_0 W_1'(0, p)]$$

$$\Delta = \left(\alpha_0\alpha_n - \frac{p^2}{a_1 a_n}\beta_0\beta_n\right)\sinh p\left(\frac{l}{a_n} + \tau_{n-1}\right) + p\left(\frac{\alpha_0\beta_n}{a_n} - \frac{\alpha_n\beta_0}{a_1}\right)\cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right) \tag{13}$$

and the formulas (8.1), (8.2) become

$$U_1(x, p) = \frac{1}{\Delta}\left\{[\gamma_0 - \alpha_0 W_1(0, p) - \beta_0 W_1'(0, p)]\left[\alpha_n \sinh p\left(\frac{l}{a_n} + \tau_{n-1} - \frac{x}{a_1}\right) + \frac{p}{a_n}\beta_n \cosh p\left(\frac{l}{a_n} + \tau_{n-1} - \frac{x}{a_1}\right)\right] + [\gamma_n - \alpha_n W_n(l, p) - \beta_n W_n'(l, p)] \cdot \left(\alpha_0 \sinh\frac{px}{a_1} - \frac{p}{a_1}\beta_0 \cosh\frac{px}{a_1}\right)\right\} + W_1(x, p), \tag{14.1}$$

$$U_{k+1}(x, p) = \frac{1}{\Delta}\left\{[\gamma_0 - \alpha_0 W_1(0, p) - \beta_0 W_1'(0, p)]\left[\alpha_n \sinh p\left(\frac{l}{a_n} - \frac{x}{a_{k+1}} + \tau_{n-1} - \tau_k\right) + \frac{p}{a_n}\beta_n \cosh p\left(\frac{l}{a_n} - \frac{x}{a_{k+1}} + \tau_{n-1} - \tau_k\right)\right] + [\gamma_n - \alpha_n W_n(l, p) - \beta_n W_n'(l, p)]\left[\alpha_0 \sinh p\left(\tau_k + \frac{x}{a_{k+1}}\right) - \frac{p}{a_1}\beta_0 \cosh p\left(\tau_k + \frac{x}{a_{k+1}}\right)\right]\right\} + W_{k+1}(x, p); \qquad k = 1, 2, \cdots, n-1. \tag{14.2}$$

It is easy to verify that the expressions (13) to (14.2) satisfy in fact the boundary conditions (7).

Formulae (14.1), (14.2) represent the Laplace transforms of $u_k(x, t)$ and from this we obtain in each concrete case the completed solution by the methods of operational calculus. Several examples are given in the following section.

4. Special problems. We illustrate the preceding theoretical considerations by typical examples generalising the results given occasionally in the literature [1]. In order to have reasonable expressions we assume (9) to be valid.

All the calculations needed in each case proceed by five stages. First, one applies the Laplace transformation to the fundamental equations governing the problem and to the initial conditions. This yields the general form of $U_k(x, p)$, as indicated in (1), and especially also the expressions $W_k(x, p)$.

When the relations (2) hold, we write down the subsidiary equations corresponding to the boundary conditions at $x = 0$, $x = l$ and obtain the values α_0 , β_0 , γ_0 , α_n , β_n , γ_n . The third and fourth steps consist in calculating the expression Δ by (13) and the Laplace transforms $U_k(x, p)$ according to (14.1), (14.2). Finally, we deduce the wanted solution $u_k(x, t)$ by the methods of operational calculus.

4.1. Vibrations of a bar under its own weight. The upper end $x = 0$ is fixed, the lower $x = l$ supported so that the displacement is zero at all points. At $t = 0$ the end $x = l$ is released.

Putting $s_0 = 0$ fundamental equations of the problem with belonging initial conditions are

$$\frac{\partial^2 u_k}{\partial t^2} = a_k^2 \frac{\partial^2 u_k}{\partial x^2} + g, \qquad s_{k-1} < x < s_k , \qquad t > 0; \qquad k = 1, 2, \cdots , n \tag{15.1}$$

$$u_k(x, 0) = 0, \qquad \frac{\partial u_k(x, 0)}{\partial t} = 0, \qquad s_{k-1} < x < s_k ; \qquad k = 1, 2, \cdots , n. \tag{15.2}$$

Solving the corresponding subsidiary equations with respect to $U_k(x, p)$ and comparing with (1) gives

$$W_k(x, p) = \frac{g}{p^2} ; \qquad k = 1, 2, \cdots , n. \tag{15.3}$$

Thus, the relations (2) hold.

The boundary conditions at $x = 0$ and $x = l$ are

$$u_1(0, t) = 0, \qquad \frac{\partial u_n(l, t)}{\partial x} = 0; \qquad t > 0 \tag{16.1}$$

and this leads to

$$U_1(0, p) = 0, \qquad U_n'(l, p) = 0.$$

We therefore have

$$\alpha_0 = \beta_n = 1, \qquad \beta_0 = \gamma_0 = \alpha_n = \gamma_n = 0 \tag{16.2}$$

and (13) gives

$$\Delta = \frac{p}{a_n} \cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right). \tag{16.3}$$

we easily obtain from (4) and (4.1)

$$M_k(p) = \begin{Vmatrix} \cosh p\sigma_k\,, & \sinh p\sigma_k \\ \sinh p\sigma_k\,, & \cosh p\sigma_k \end{Vmatrix}; \qquad k = 1, 2, \cdots, n-1$$
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However,

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so that (4.1) yields the closed forms

$$Q_k(p) = \begin{Vmatrix} \cosh p\tau_k\,, & \sinh p\tau_k \\ \sinh p\tau_k\,, & \cosh p\tau_k \end{Vmatrix}; \qquad k = 1, 2, \cdots, n-1 \tag{12}$$

of the fundamental matrices $Q_k(p)$.

Equations (8.3) then give

$$\Delta_1 = \left[\alpha_n \sinh p\left(\frac{l}{a_n} + \tau_{n-1}\right) + \frac{p}{a_n}\beta_n \cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right)\right][\gamma_0 - \alpha_0 W_1(0, p) - \beta_0 W_1'(0, p)] - \frac{p}{a_1}\beta_0[\gamma_n - \alpha_n W_n(l, p) - \beta_n W_n'(l, p)]$$

$$\Delta_2 = \alpha_0[\gamma_n - \alpha_n W_n(l, p) - \beta_n W_n'(l, p)] - \left[\alpha_n \cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right) + \frac{p}{a_n}\beta_n \sinh p\left(\frac{l}{a_n} + \tau_{n-1}\right)\right][\gamma_0 - \alpha_0 W_1(0, p) - \beta_0 W_1'(0, p)]$$

$$\Delta = \left(\alpha_0\alpha_n - \frac{p^2}{a_1 a_n}\beta_0\beta_n\right)\sinh p\left(\frac{l}{a_n} + \tau_{n-1}\right) + p\left(\frac{\alpha_0\beta_n}{a_n} - \frac{\alpha_n\beta_0}{a_1}\right)\cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right) \tag{13}$$

and the formulas (8.1), (8.2) become

$$U_1(x, p) = \frac{1}{\Delta}\left\{[\gamma_0 - \alpha_0 W_1(0, p) - \beta_0 W_1'(0, p)]\left[\alpha_n \sinh p\left(\frac{l}{a_n} + \tau_{n-1} - \frac{x}{a_1}\right) + \frac{p}{a_n}\beta_n \cosh p\left(\frac{l}{a_n} + \tau_{n-1} - \frac{x}{a_1}\right)\right] + [\gamma_n - \alpha_n W_n(l, p) - \beta_n W_n'(l, p)] \cdot \left(\alpha_0 \sinh\frac{px}{a_1} - \frac{p}{a_1}\beta_0 \cosh\frac{px}{a_1}\right)\right\} + W_1(x, p), \tag{14.1}$$

$$U_{k+1}(x, p) = \frac{1}{\Delta}\left\{[\gamma_0 - \alpha_0 W_1(0, p) - \beta_0 W_1'(0, p)]\left[\alpha_n \sinh p\left(\frac{l}{a_n} - \frac{x}{a_{k+1}} + \tau_{n-1} - \tau_k\right) + \frac{p}{a_n}\beta_n \cosh p\left(\frac{l}{a_n} - \frac{x}{a_{k+1}} + \tau_{n-1} - \tau_k\right)\right] + [\gamma_n - \alpha_n W_n(l, p) - \beta_n W_n'(l, p)]\left[\alpha_0 \sinh p\left(\tau_k + \frac{x}{a_{k+1}}\right) - \frac{p}{a_1}\beta_0 \cosh p\left(\tau_k + \frac{x}{a_{k+1}}\right)\right]\right\} + W_{k+1}(x, p); \qquad k = 1, 2, \cdots, n-1. \tag{14.2}$$

It is easy to verify that the expressions (13) to (14.2) satisfy in fact the boundary conditions (7).

Formulae (14.1), (14.2) represent the Laplace transforms of $u_k(x, t)$ and from this we obtain in each concrete case the completed solution by the methods of operational calculus. Several examples are given in the following section.

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All the calculations needed in each case proceed by five stages. First, one applies the Laplace transformation to the fundamental equations governing the problem and to the initial conditions. This yields the general form of $U_k(x, p)$, as indicated in (1), and especially also the expressions $W_k(x, p)$.

When the relations (2) hold, we write down the subsidiary equations corresponding to the boundary conditions at $x = 0$, $x = l$ and obtain the values α_0 , β_0 , γ_0 , α_n , β_n , γ_n . The third and fourth steps consist in calculating the expression Δ by (13) and the Laplace transforms $U_k(x, p)$ according to (14.1), (14.2). Finally, we deduce the wanted solution $u_k(x, t)$ by the methods of operational calculus.

4.1. Vibrations of a bar under its own weight. The upper end $x = 0$ is fixed, the lower $x = l$ supported so that the displacement is zero at all points. At $t = 0$ the end $x = l$ is released.

Putting $s_0 = 0$ fundamental equations of the problem with belonging initial conditions are

$$\frac{\partial^2 u_k}{\partial t^2} = a_k^2 \frac{\partial^2 u_k}{\partial x^2} + g, \qquad s_{k-1} < x < s_k , \qquad t > 0; \qquad k = 1, 2, \cdots , n \tag{15.1}$$

$$u_k(x, 0) = 0, \qquad \frac{\partial u_k(x, 0)}{\partial t} = 0, \qquad s_{k-1} < x < s_k ; \qquad k = 1, 2, \cdots , n. \tag{15.2}$$

Solving the corresponding subsidiary equations with respect to $U_k(x, p)$ and comparing with (1) gives

$$W_k(x, p) = \frac{g}{p^2} ; \qquad k = 1, 2, \cdots , n. \tag{15.3}$$

Thus, the relations (2) hold.

The boundary conditions at $x = 0$ and $x = l$ are

$$u_1(0, t) = 0, \qquad \frac{\partial u_n(l, t)}{\partial x} = 0; \qquad t > 0 \tag{16.1}$$

and this leads to

$$U_1(0, p) = 0, \qquad U_n'(l, p) = 0.$$

We therefore have

$$\alpha_0 = \beta_n = 1, \qquad \beta_0 = \gamma_0 = \alpha_n = \gamma_n = 0 \tag{16.2}$$

and (13) gives

$$\Delta = \frac{p}{a_n} \cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right). \tag{16.3}$$

It then follows from (14.1), (14.2) that

$$U_1(x, p) = \frac{g}{p^2}\left[1 - \frac{\cosh p\left(\frac{l}{a_n} + \tau_{n-1} - \frac{x}{a_1}\right)}{\cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right)}\right],$$

$$U_{k+1}(x, p) = \frac{g}{p^2}\left[1 - \frac{\cosh p\left(\frac{l}{a_n} - \frac{x}{a_{k+1}} + \tau_{n-1} - \tau_k\right)}{\cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right)}\right]; \quad k = 1, 2, \cdots, n-1 \quad (17.1)$$

and from this we find (with the aid of a table of Laplace transforms or by contour integration) the complete solution in the form

$$u_1(x, t) = \frac{gx}{a_1}\left(\frac{l}{a_n} + \tau_{n-1} - \frac{x}{2a_1}\right) - \frac{16g}{\pi^3}\left(\frac{l}{a_n} + \tau_{n-1}\right)^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cos \frac{(2n+1)\pi\left(\frac{l}{a_n} + \tau_{n-1} - \frac{x}{a_1}\right)}{2\left(\frac{l}{a_n} + \tau_{n-1}\right)} \cdot \cos \frac{(2n+1)\pi t}{2\left(\frac{l}{a_n} + \tau_{n-1}\right)}$$

$$u_{k+1}(x, t) = g\left(\frac{x}{a_{k+1}} + \tau_k\right)\left(\frac{l}{a_n} + \tau_{n-1} - \frac{x}{2a_{k+1}} - \frac{\tau_k}{2}\right) - \frac{16g}{\pi^3}\left(\frac{l}{a_n} + \tau_{n-1}\right)^2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cos \frac{(2n+1)\pi\left(\frac{l}{a_n} + \tau_{n-1} - \frac{x}{a_{k+1}} - \tau_k\right)}{2\left(\frac{l}{a_n} + \tau_{n-1}\right)} \cos \frac{(2n+1)\pi t}{2\left(\frac{l}{a_n} + \tau_{n-1}\right)};$$

$$k = 1, 2, \cdots, n-1. \quad (17.2)$$

4.2. Bar with a mass at the end. The end $x = 0$ is fixed, a mass m is attached to the other end $x = l$. The bar is initially stretched by the axial force qS and at $t = 0$ its end $x = l$ is released.

The fundamental equations

$$\frac{\partial^2 u_k}{\partial t^2} = a_k^2 \frac{\partial^2 u_k}{\partial x^2}, \qquad s_{k-1} < x < s_k, \qquad t > 0; \qquad k = 1, 2, \cdots, n \quad (18.1)$$

with

$$u_k(x, 0) = S\left(\sum_{\varkappa=1}^{k-1} \frac{l_\varkappa}{E_\varkappa} + \frac{x - s_{k-1}}{E_k}\right), \qquad \frac{\partial u_k(x, 0)}{\partial t} = 0, \qquad s_{k-1} < x < s_k;$$

$$k = 1, 2, \cdots, n \quad (18.2)$$

and with the boundary conditions

$u_1(0, t) = 0$

$$m\frac{\partial^2 u_n(l,t)}{\partial t^2} = -qE_n\frac{\partial u_n(l,t)}{\partial x}, \quad u_n(l,0) = S\sum_{\kappa=1}^{n}\frac{l_\kappa}{E_\kappa}, \quad \frac{\partial u_n(l,0)}{\partial t} = 0; \quad t > 0 \tag{18.3}$$

give, as in the preceding case

$$W_k(x,p) = S\left(\sum_{\kappa=1}^{k-1}\frac{l_\kappa}{E_\kappa} + \frac{x - s_{k-1}}{E_k}\right), \qquad \alpha_0 = 1, \qquad \beta_0 = \gamma_0 = 0,$$
$$\alpha_n = 1, \qquad \beta_n = \frac{qE_n}{mp^2}, \qquad \gamma_n = S\sum_{\kappa=1}^{n}\frac{l_\kappa}{E_\kappa}. \tag{19}$$

Thus, we find without difficulty

$$\Delta = \sinh p\left(\frac{l}{a_n} + \tau_{n-1}\right) + \frac{qE_n}{mpa_n}\cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right) \tag{20}$$

and formulae (14.1), (14.2) yield

$$U_1(x,p) = \frac{Sx}{E_1} - \frac{qS}{m}\cdot\frac{\sinh\dfrac{px}{a_1}}{p\left[p\sinh p\left(\dfrac{l}{a_n} + \tau_{n-1}\right) + \dfrac{qE_n}{ma_n}\cosh p\left(\dfrac{l}{a_n} + \tau_{n-1}\right)\right]}$$

$$U_{k+1}(x,p) = S\left(\sum_{\kappa=1}^{k}\frac{l_\kappa}{E_\kappa} + \frac{x - s_k}{E_{k+1}}\right) \tag{21.1}$$
$$- \frac{qS}{m}\cdot\frac{\sinh p\left(\tau_k + \dfrac{x}{a_{k+1}}\right)}{p\left[p\sinh p\left(\dfrac{l}{a_n} + \tau_{n-1}\right) + \dfrac{qE_n}{ma_n}\cosh p\left(\dfrac{l}{a_n} + \tau_{n-1}\right)\right]};$$
$$k = 1, 2, \cdots, n-1.$$

The solution is

$$u_1(x,t) = \frac{Sx}{E_1}\left(1 - \frac{a_n}{a_1}\frac{E_1}{E_n}\right) + \frac{2qS}{m}\left(\frac{l}{a_n} + \tau_{n-1}\right)^2$$
$$\cdot\sum_{\rho=1}^{\infty}\frac{\sec\delta_\rho}{\delta_\rho\left[\left(\dfrac{l}{a_n} + \tau_{n-1}\right)\dfrac{qE_n}{ma_n} + \left(\dfrac{l}{a_n} + \tau_{n-1}\right)^2\left(\dfrac{qE_n}{ma_n}\right)^2 + \delta_\rho^2\right]}\sin\frac{\dfrac{\delta_\rho x}{a_1}}{\dfrac{l}{a_n} + \tau_{n-1}}\cos\frac{\delta_\rho t}{\dfrac{l}{a_n} + \tau_{n-1}}$$

$$u_{k+1}(x,t) = S\left[\sum_{\kappa=1}^{k}\frac{l_\kappa}{E_\kappa} - \frac{s_k}{E_{k+1}} - \frac{a_n\tau_k}{E_n} + \frac{x}{E_{k+1}}\left(1 - \frac{a_n}{a_{k+1}}\frac{E_{k+1}}{E_n}\right)\right]$$
$$+ \frac{2qS}{m}\left(\frac{l}{a_n} + \tau_{n-1}\right)^2\sum_{\rho=1}^{\infty}\frac{\sec\delta_\rho}{\delta_\rho\left[\left(\dfrac{l}{a_n} + \tau_{n-1}\right)\dfrac{qE_n}{ma_n} + \left(\dfrac{l}{a_n} + \tau_{n-1}\right)^2\left(\dfrac{qE_n}{ma_n}\right)^2 + \delta_\rho^2\right]}$$
$$\cdot\sin\frac{\delta_\rho\left(\tau_k + \dfrac{x}{a_{k+1}}\right)}{\dfrac{l}{a_n} + \tau_{n-1}}\cos\frac{\delta_\rho t}{\dfrac{l}{a_n} + \tau_{n-1}}; \qquad k = 1, 2, \cdots, n-1. \tag{21.2}$$

The $\delta_\rho (\rho = 1, 2, \cdots)$ are the positive roots of

$$\delta \tan \delta = \left(\frac{l}{a_n} + \tau_{n-1}\right) \frac{qE_n}{ma_n}. \tag{21.3}$$

4.3. Collision of two rods. Two equal rods of the kind treated above move longitudinally in opposite directions with equal speeds v and collide at $t = 0$, at the origin $x = 0$. By symmetry we consider only the rod $0 < x < l$ and so long as the rods are in contact we have the equations

$$\frac{\partial^2 u_k}{\partial t^2} = a_k^2 \frac{\partial^2 u_k}{\partial x^2}, \qquad s_{k-1} < x < s_k, \qquad t > 0; \qquad k = 1, 2, \cdots, n \tag{22.1}$$

with the initial conditions

$$u_k(x, 0) = 0, \qquad \frac{\partial u_k(x, 0)}{\partial t} = -v, \qquad s_{k-1} < x < s_k; \qquad k = 1, 2, \cdots, n \tag{22.2}$$

and with

$$u_1(0, t) = 0, \qquad \frac{\partial u_n(l, t)}{\partial x} = 0, \qquad t > 0. \tag{22.3}$$

From this we obtain, as before

$$W_k(x, p) = -\frac{v}{p} (k = 1, 2, \cdots, n); \quad \alpha_0 = \beta_n = 1, \quad \beta_0 = \gamma_0 = \alpha_n = \gamma_n = 0 \tag{23.1}$$

$$\Delta = \frac{p}{a_n} \cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right), \tag{23.2}$$

and formulae (14.1), (14.2) become

$$U_1(x, p) = -\frac{v}{p}\left[1 - \frac{\cosh p\left(\frac{l}{a_n} + \tau_{n-1} - \frac{x}{a_1}\right)}{\cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right)}\right]$$

$$U_{k+1}(x, p) = -\frac{v}{p}\left[1 - \frac{\cosh p\left(\frac{l}{a_n} + \tau_{n-1} - \tau_k - \frac{x}{a_{k+1}}\right)}{\cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right)}\right];$$

$$k = 1, 2, \cdots, n - 1. \tag{24.1}$$

Hence, the solution is

$$u_1(x, t) = \frac{8v}{\pi^2}\left(\frac{l}{a_n} + \tau_{n-1}\right) \cdot \sum_{r=0}^{\infty} \frac{(-1)^{r+1}}{(2r+1)^2} \cos \frac{(2r+1)\pi\left(\frac{l}{a_n} + \tau_{n-1} - \frac{x}{a_1}\right)}{2\left(\frac{l}{a_n} + \tau_{n-1}\right)} \sin \frac{(2r+1)\pi t}{2\left(\frac{l}{a_n} + \tau_{n-1}\right)}$$

$$u_{k+1}(x, t) = \frac{8v}{\pi^2}\left(\frac{l}{a_n} + \tau_{n-1}\right)$$

$$\cdot \sum_{r=0}^{\infty} \frac{(-1)^{r+1}}{(2r+1)^2} \cos \frac{(2r+1)\pi\left(\frac{l}{a_n} + \tau_{n-1} - \tau_k - \frac{x}{a_{k+1}}\right)}{2\left(\frac{l}{a_n} + \tau_{n-1}\right)} \sin \frac{(2r+1)\pi t}{2\left(\frac{l}{a_n} + \tau_{n-1}\right)};$$

$$k = 1, 2, \cdots, n - 1. \tag{24.2}$$

5. Final remarks. For $n = 1$ the formulae for $u_1(x, t)$ in (17.2), (21.2) and (24.2) change into the well-known elementary results given in the literature [1].

Our preceding deductions may serve as a kind of guide in treating other problems on vibrations of composite continua. A special example has been discussed in detail in the author's paper given in the literature [2]. The method of matrix analysis can be used also in treating other important questions of engineering such as torsional oscillations [3], conduction of heat [4, 5], etc.

References

1. H. S. Carslaw and J. C. Jaeger, *Operational methods in applied mathematics*, 2nd ed., Oxford University Press, 1947
2. V. Vodička, Longitudinal vibrations of a composite bar, ZAMP **7**, 345 (1956)
3. L. A. Pipes, Electrical circuit analysis of torsional oscillations, J. Appl. Phys. **14**, 352 (1943)
4. V. Vodička, Conduction of fluctuating heat flow in a wall consisting of many layers, Appl. Sci. Research **A5**, 108 (1955)
5. V. Vodička, Heat waves in multilayer cylindrical bodies, Appl. Sci. Research **A5**, 115 (1955)

BOOK REVIEWS

Plane waves and spherical means applied to partial differential equations. By Fritz John. Interscience Publishers, Inc., New York, 1955. viii + 172 pp. $4.50.

The present tract contains a collection of various results on partial differential equations, their unifying thread being the use of certain elementary identities for plane and spherical integrals of an arbitrary function. The author's principal aim is to show that many results on fairly general differential equations follow from those identities. The book begins with a concise introduction, which illuminates the paths to be followed. Chapter I deals with the decomposition of arbitrary functions into functions of the type of plane waves, i.e. into functions having parallel planes as level surfaces. The decomposition used here consists of expressing a function by spherical means of integrals over hyperplanes, and is due to J. Radon. This kind of decomposition is to be compared with that furnished by Fourier analysis (into plane waves of exponential type). The "Randon transform" has the advantage over the "Fourier transform" that the integrals appearing in it involve the given function itself, and not its Fourier transform. In Chapter II the Randon transform is applied to the solution of the Cauchy problem for the homogeneous hyperbolic equation with constant coefficients $L(u) = Q(\partial/\partial x_1 , \cdots , \partial/\partial x_n , \partial/\partial t)\ u = 0$, where $Q(\eta_1 , \cdots , \eta_n , \lambda)$ is a form of degree m in its arguments, with constant real coefficients; with the initial data $(\partial^k u/\partial t^k)_{t=0} = f_k(x_1 , \cdots , x_n)$ for $k = 0, 1, \cdots , m - 1$. Chapter III gives a construction of a fundamental solution for a linear elliptic equation, and for a system, with analytic coefficients. The problem amounts to finding a solution of the symbolic equation $L(u) = \delta$, where δ is the Dirac function; and the method of solution employed is that of decomposing the δ function into plane wave functions, thus reducing the problem to that of solving $L(u) = f$, where f is a plane wave function. In Chapter IV are found formulas for an arbitrary function in terms of spherical integrals of the function. These formulas, which form the principal tool used in the later chapters, can be regarded as generalizations of the expressions of Chapter I, which give the arbitrary function in terms of its integrals over planes. They can also be viewed "as the analytic counterpart of the geometrical fact that spherical shells can be swept out by spheres in two different ways, just as the identities of Chapter I are connected with the fact that the exterior of a sphere can be swept out by planes". Chapter V is concerned with the mean value theorem of Asgeirsson for the ultra-hyperbolic equation, together with a somewhat more general identity, having to do with the ellipsoidal means of a function, due to A. S. Howard. Chapter V deals mostly with the problem of the determination of a function from a knowledge of its integrals over a sphere of fixed radius, which can be solved by means of the identities of Chapter IV. Chapter VII contains the major application of the identities on spherical means in Chapter IV, to the proof of the differentiability of solutions of linear or non-linear elliptic equations or systems, provided that the coefficients are sufficiently regular. Chapter VIII extends the results of the preceding chapter to non-linear elliptic equations, save that here the regularity not of the solutions themselves, but that of certain integral transforms of them is established. The lucidity of the exposition combines with the attractive format of the book to make it the delight of mathematicians interested in the subject.

J. B. Diaz

Recent advances in science (Physics and Applied Mathematics). Edited by M. H. Shamos and G. M. Murphy. New York University Press, 1956. xi + 384 pp. $7.50.

The book is an outgrowth of the First Symposium on Recent Advances in Science held at New York University in the spring of 1954; it contains the following articles: Methods of Applied Mathematics (R. Courant); The Future of Operations Research (P. M. Morse); Atomic Structure (I.I. Rabi); Microwave Spectroscopy (C. H. Townes); Nuclear Structure and Transmutations (H. A. Bethe); Elementary Particles (V. F. Weisskopf); Electronuclear Machines (L. J. Haworth); Neutron Physics (N. F. Ramsey); Transistor Physics (W. Shockley); Ferromagnetism (R. M. Bozorth); Cryogenics: Very-Low Temperature Physics and Engineering (F. G. Brickwedde); Physics and the Engineer (E. U. Condon).

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APPROXIMATE SOLUTIONS FOR A CLASS OF INTEGRAL EQUATIONS*

BY

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1. Introduction. Exact solutions for integral equations of the type

$$f(x) = g(x) + \lambda \int_0^a dy\, k(x - y) f(y), \qquad 0 \leq x \leq a, \tag{1}$$

are, in general, exceedingly difficult to obtain, except for particularly simple forms for $k(x - y)$ such as a sum of polynomials times exponentials. In the present discussion, it will be shown that by an extension of the Wiener-Hopf technique, it is possible to obtain approximate solutions as well as approximate eigenvalues for this equation. The accuracy of the solutions will be shown to improve exponentially with increasing order of approximation. Application of the method is made to the cases wherein $k(x) = e^{-|x|}$, for which case an exact solution is obtained, and $k(x) = 1/(2\pi)^{1/2} \exp(-1/2x^2)$, for which case the lowest eigenvalue is obtained approximately.

2. Fourier transform of (1). The first step in the present method consists in obtaining the generalized Fourier transform (Titchmarsh [1]) of (1). For this purpose, it is assumed that $f(x) = 0(e^{c|x|})$ $(c < 1)$, $g(x) = 0(e^{d|x|})$ $(d < 1)$, $k(x) = 0(e^{-|x|})$, and $k(x) = k(-x)$, $k(x)$ real; and that these functions belong to $L_2(-\infty, \infty)$. The following arguments treat (1) as if it holds for all x. This constitutes no limitation since if (1) is assumed to apply only for $0 \leq x \leq a$, then the conditions $k(x) = 0$ for $|x| > a$ and $g(x) = 0$ for $x < -a$ and for $x > 2a$ may be arbitrarily required. From these conditions it is observed that $f(x) = 0$ for $x < -a$ and for $x > 2a$ and therefore $f(x)$, $g(x)$ and $k(x)$ automatically satisfy the assumed growth conditions. To simplify the subsequent formulas, the following quantities are defined and their region of analyticity indicated:

$$(i) \qquad f_+(w) = \int_a^\infty dx\, e^{ixw} f(x) \quad (\mathrm{Im}\, w > c),$$

$$(ii) \qquad f_a(w) = \int_0^a dx\, e^{ixw} f(x) \quad (\mathrm{Im}\, w > -\infty),$$

$$(iii) \qquad f_-(w) = \int_{-\infty}^0 dx\, e^{ixw} f(x) \quad (\mathrm{Im}\, w < -c),$$

$$(iv) \qquad K(w) = \int_{-\infty}^\infty dx\, e^{ixw} k(x) \quad (-1 < \mathrm{Im}\, w < 1),$$

$$(v) \qquad g_+(w) = \int_a^\infty dx\, e^{ixw} g(x) \quad (\mathrm{Im}\, w > d),$$

$$(vi) \qquad g_a(w) = \int_0^a dx\, e^{ixw} g(x) \quad (\mathrm{Im}\, w > -\infty),$$

$$(vii) \qquad g_-(w) = \int_{-\infty}^0 dx\, e^{ixw} g(x) \quad (\mathrm{Im}\, w < -d).$$

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In terms of these quantities and by the Fourier inversion formula, (1) becomes

$$\int_{i\mu-\infty}^{i\mu+\infty} dw\, e^{-izw}[f_-(w) - g_-(w)] + \int_{i\tau-\infty}^{i\tau+\infty} dw\, e^{-izw}[f_+(w) - g_+(w) - g_a(w) - (\lambda K(w) - 1)f_a(w)] = 0, \tag{2}$$

where $\tau > -1$; $c, d < \tau < 1$, and $\mu < -c, -d$. By a theorem due to Titchmarsh ([1], p. 255), (2) yields directly for the transform of (1)

$$[\lambda K(w) - 1]f_a(w) = f_+(w) + f_-(w) - g_+(w) - g_-(w) - g_a(w), \tag{3}$$

where in the region $|\operatorname{Im} w| < 1$, the functions $f_-(w) - g_-(w)$ and $f_+(w) - g_+(w) - g_a(w) - (\lambda K(w) - 1)\, f_a(w)$ are regular and tend to zero uniformly as $|\operatorname{Rl} w| \to \infty$ in any interior strip. Since $f_a(w)$, $g_a(w)$ and $K(w)$ are also regular and tend to zero uniformly as $|\operatorname{Rl} w| \to \infty$ in any strip in $|\operatorname{Im} w| < 1$, the same properties hold for $f_+(w) - g_+(w)$.

3. Integral equations for $f_+(w)$ and $f_-(w)$. In order to evaluate the functions $f_a(w)$, $f_-(w)$ and $f_+(w)$ which determine the solution of (1), we shall need the following easily proved properties:

(i) $f_a(w)$ and $g_a(w)$ are entire functions of w,
(ii) $f_+(w) - g_+(w)$ is regular for $\operatorname{Im} w > -1$,
(iii) $f_-(w) - g_-(w)$ is regular for $\operatorname{Im} w < 1$,
(iv) $\lambda K(w) - 1 = \dfrac{\chi_-(w)}{\chi_+(w)}\,\sigma(w)$ for $-1 < -\alpha \leq \alpha < 1$ ([1], p. 339 and [2], p. 51)

where

(a) $\chi_-(w)$ is regular and free of zeros for $\operatorname{Im} w \leq \alpha$,
(b) $m_- < |\chi_-(w)w^{n/2}| < M_-$ for $\operatorname{Im} w \leq \alpha$, where m_- and M_- are positive constants,
(c) $\chi_+(w)$ is regular and free of zeros for $\operatorname{Im} w \geq -\alpha$,
(d) $m_+ < |\chi_+(w)\, w^{-n/2}| < M_+$ for $\operatorname{Im} w \geq -\alpha$, where m_+ and M_+ are positive constants,
(e) $\sigma(w) = \epsilon(w)\epsilon(-w)$ and $\epsilon(w) = \prod_{i=1}^{n/2}(w - w_i)$, where n is the number of roots w_i and $-w_i$ of $\lambda\, K(w) - 1$ in $|\operatorname{Im} w| \leq \alpha$.

Using the property (iv), we may rewrite (3) as

$$\frac{\chi_-(w)}{\chi_+(w)}\,\epsilon(w)\epsilon(-w)f_a(w) = f_+(w) + f_-(w) - g_+(w) - g_-(w) - g_a(w). \tag{4}$$

The present method depends upon converting (4) into two simultaneous integral equations for $f_+(w)$ and $f_-(w)$. To obtain the first of these equations, we re-express (4) as

$$f_a(w)\,\frac{\epsilon(-w)}{\chi_+(w)} = \frac{f_+(w) - g_+(w) - g_a(w)}{\epsilon(w)\chi_-(w)} + \frac{f_-(w) - g_-(w)}{\epsilon(w)\chi_-(w)}, \tag{4'}$$

where from the properties (i), (iii) and (iv) the term on the left-hand side is regular for $\operatorname{Im} w > -\alpha$ and the second term on the right-hand side is regular except at the zeros of $\epsilon(w)$ for $\operatorname{Im} w < \alpha$. We now consider a contour C_α which connects the points

$i\alpha - \infty$ and $i\alpha + \infty$ by a curve which lies in the region $-\alpha < \text{Im } w < \alpha$ and passes below the zeros of $\epsilon(w)$ and a contour $C_{-\alpha}$ which connects the points $-i\alpha - \infty$ and $-i\alpha + \infty$ by a curve lying below C_α but above Im $w = -\alpha$. Then we split the term

$$\frac{f_+(w) - g_+(w) - g_\alpha(w)}{\epsilon(w)\chi_-(w)},$$

and hence the whole of equation (4'), into two parts, one regular in the half-plane above $C_{-\alpha}$ and the other below C_α. Thus by Cauchy's theorem

$$\frac{f_+(w) - g_+(w) - g_\alpha(w)}{\epsilon(w)\chi_-(w)} = \frac{1}{2\pi i}\left\{\int_{C_{-\alpha}} - \int_{C_\alpha} + \int_{-i\alpha+\infty}^{i\alpha+\infty} + \int_{i\alpha-\infty}^{-i\alpha-\infty}\right\} \cdot \frac{dw'}{w' - w} \frac{f_+(w') - g_+(w') - g_\alpha(w')}{\epsilon(w')\chi_-(w')}$$

where w lies within the closed contour. From the remark following (3) it follows that the third and fourth integrals on the right-hand side of this equation vanish. Thus

$$\begin{aligned}\frac{f_+(w) - g_+(w) - g_\alpha(w)}{\epsilon(w)\chi_-(w)} &= \frac{1}{2\pi i}\left\{\int_{C_{-\alpha}} - \int_{C_\alpha}\right\} \frac{dw'}{w' - w} \frac{f_+(w') - g_+(w') - g_\alpha(w')}{\epsilon(w')\chi_-(w')} \\ &= I(w, C_{-\alpha}) - I(w, C_\alpha).\end{aligned} \tag{5}$$

From the definition of $I(w, C_{-\alpha})$, we see that this function is regular for Im w above $C_{-\alpha}$ which results from the uniform convergence of the integral with respect to w in any region whose closure is in the region Im w above $C_{-\alpha}$. Similarly, $I(w, C_\alpha)$ is regular for Im w in the half plane below C_α. Thus (4') may be written

$$\frac{f_-(w) - g_-(w)}{\epsilon(w)\chi_-(w)} - I(w, C_\alpha) = f_\alpha(w)\frac{\epsilon(-w)}{\chi_+(w)} - I(w, C_{-\alpha}). \tag{6}$$

From the preceding remarks, the right-hand side of this equation is regular for Im w above $C_{-\alpha}$ and the left-hand side is regular for Im w below C_α. Thus (6) defines a function regular in the whole w-plane and thus it defines an entire function. From properties (iv) (b) and (d), $1/\epsilon(w)\chi_-(w)$ and $\epsilon(-w)/\chi_+(w)$ are bounded for Im w below C_α and above $C_{-\alpha}$, respectively. Further, from Theorems III and V of [2], $f_-(w) - g_-(w)$ is bounded in every properly included half-plane below C_α. Analogously, $f_\alpha(w)$ is bounded in the half-plane above $C_{-\alpha}$. Finally, from the uniform convergence of $I(w, C_{-\alpha})$ and $I(w, C_\alpha)$ with respect to w, it follows that these quantities are bounded and approach zero as $|w| \to \infty$ in the interior of their respective half-planes of regularity. Thus, the two sides of (6) are bounded and, hence, (6) defines a polynomial $p(w)$ of degree at most zero. However, since $f_-(w) - g_-(w)$, $f_\alpha(w)$, $I(w, C_{-\alpha})$, and $I(w, C_\alpha)$ vanish as $|\text{Rl } w| \to \infty$ in their respective regions of regularity, $p(w)$ must be identically zero. Thus

$$f_\alpha(w) = \frac{\chi_+(w)}{\epsilon(-w)} \frac{1}{2\pi i} \int_{C_{-\alpha}} \frac{dw'}{w' - w} \frac{f_+(w') - g_+(w') - g_\alpha(w')}{\epsilon(w')\chi_-(w')} \tag{7}$$

for Im w above $C_{-\alpha}$ and

$$\frac{f_-(w) - g_-(w)}{\epsilon(w)\chi_-(w)} = \frac{1}{2\pi i} \int_{C_\alpha} \frac{dw'}{w' - w} \frac{f_+(w') - g_+(w') - g_\alpha(w')}{\epsilon(w')\chi_-(w')} \tag{8}$$

for Im w below C_α.

In analogy with the calculation just described, if we start with (4), rewriting it as

$$\chi_-(w)\epsilon(w)e^{-iaw}f_a(w) = \frac{\chi_+(w)}{\epsilon(-w)}e^{-iaw}[f_+(w) - g_+(w)] + \frac{\chi_+(w)}{\epsilon(-w)}e^{-iaw}[f_-(w) - g_-(w) - g_a(w)], \tag{4''}$$

then split the second term on the right-hand side into two parts, one regular for Im w below C_β and the other regular for Im w above $C_{-\beta}$, where $C_{-\beta}$ connects $-i\beta - \infty$ to $-i\beta + \infty$ by a contour passing above the zeros of $\epsilon(-w)$ and contained in the region $-\beta$ Im $w < \beta$, and C_β connects $i\beta - \infty$ to $i\beta + \infty$ and lies above $C_{-\beta}$ in the strip $-\beta <$ Im $w < \beta$, then we have

$$f_a(w) = -\frac{e^{iaw}}{\chi_-(w)\epsilon(w)}\frac{1}{2\pi i}\int_{C_\beta}\frac{dw'}{w' - w}\frac{f_-(w') - g_-(w') - g_a(w')}{\epsilon(-w')}\chi_+(w')e^{-iaw'} \tag{9}$$

for Im w below C_β and

$$[f_+(w) - g_+(w)]\frac{\chi_+(w)}{\epsilon(-w)}e^{-iaw} = -\frac{1}{2\pi i}\int_{C_{-\beta}} \cdot\frac{dw'}{w' - w}\frac{f_-(w') - g_-(w') - g_a(w')}{\epsilon(-w')}\chi_+(w')e^{-iaw'} \tag{10}$$

for Im w above $C_{-\beta}$. The presence of e^{-iaw} and the specific arrangement of $g_a(W)$ in (4'') are required in order to bound the two sides of (4'') according to the argument presented after (6); in particular, for this purpose, it is necessary to have the easily derived results that $|\, e^{-iaw}f_a(w)\,| = o(1)$ as Im $w \to -\infty$ and $|\, e^{-iaw}[f_+(w) - g_+(w)]\,| = o(1)$ as Im $w \to \infty$.

The important result is the pair of integral equations (8) and (10). The solutions of these equations, when combined with (7), determine $f_a(w)$ and finally $f(x)$. Unfortunately, exact solutions can only be obtained for the case that $K(w)$ is a rational fraction in w. However, approximate solutions can be readily obtained.

4. Approximate solutions of (8) and (10). To solve (8) and (10) approximately we let the contour C_α be shifted toward $+i\infty$ and the contour $C_{-\beta}$ be shifted toward $-i\infty$. The contours sweep across the singularities of the integrands of (8) and (10). The singularities are easily seen to be poles, except possibly for essential singularities arising from $\chi_+(w)$ and $\chi_-(w)$. If such essential singularities exist, they may be removed from the finite part of the plane by a simple device. Namely, since only values of $k(x)$ in the interval $-a \le x \le a$ enter into (1), $k(x)$ may be chosen zero outside this interval as remarked in Sec. 2. As a consequence, $K(w)$ becomes an entire function and, therefore, $\chi_+(w)$ and $\chi_-(w)$ have no essential singularities in the finite part of the w-plane. Thus if the contours are moved to infinity, the only singularities crossed will be poles. In this manner, (8) and (10) may be written

$$\frac{f_-(w) - g_-(w)}{\epsilon(w)\chi_-(w)} = r(w) + \sum \text{Res.}\left(\frac{f_+(w')}{\epsilon(w')\chi_-(w')}\frac{1}{w' - w}\right) + R(w, \alpha'), \tag{11}$$

$$[f_+(w) - g_+(w)]\frac{\chi_+(w)}{\epsilon(-w)}e^{-iaw} = s(w) + \sum \text{Res.}\left(\frac{f_-(w')\chi_+(w')}{\epsilon(-w')}\frac{e^{-iaw'}}{w' - w}\right) + S(w, \beta'), \tag{12}$$

where

$$r(w) = -\frac{1}{2\pi i}\int_{C_\alpha} \frac{dw'}{w'-w}\,\frac{g_+(w')+g_a(w')}{\epsilon(w')\chi_-(w')},$$

$$s(w) = \frac{1}{2\pi i}\int_{C_{-\beta}} \frac{dw'}{w'-w}\,\frac{[g_-(w')+g_a(w')]\chi_+(w')}{\epsilon(-w')}\,e^{-iaw'},$$

$$R(w,\alpha') = \frac{1}{2\pi i}\int_{i\alpha'-\infty}^{i\alpha'+\infty} \frac{dw'}{w'-w}\,\frac{f_+(w')}{\epsilon(w')\chi_-(w')},$$

$$S(w,\beta') = -\frac{1}{2\pi i}\int_{-i\beta'-\infty}^{-i\beta'+\infty} \frac{dw'}{w'-w}\,\frac{f_-(w')\chi_+(w')}{\epsilon(-w')}\,e^{-iaw'}.$$

In (11) and (12) the summations are over the residues at the poles of the integrands crossed by the contours as C_α is moved up to $i\alpha'$ and $C_{-\beta}$ is moved down to $-i\beta'$. Approximate solutions of (8) and (10) are therefore provided by (11) and (12) if $R(w, \alpha')$ and $S(w, \beta')$ are neglected. The residues in (11) and (12) are obtained in this approximation in terms of the approximation itself by solving an obvious system of linear equations derived from (11) and (12). In the case that the contours C_α and $C_{-\beta}$ sweep across multiple poles, the residues of (11) and (12) involve derivatives of $f_+(w)$ and $f_-(w)$. These derivatives may be evaluated by differentiating (11) and (12), and including the resultant equations into an extended system of linear equations for determining the residues.

A question may be raised with regard to the uniqueness and convergence of the residues obtained from the solution of the linear system derived from (11) and (12). It has not been possible as yet to establish rigorously and generally this uniqueness and convergence. In a number of simple applications of the present method, in particular to cases wherein $K(w)$ is a rational fraction, the linear system has been found to have unique solutions; that is, the determinant of the system was non-singular except at eigenvalues. Moreover, for $K(w)$ a rational fraction, convergence was automatic, since the determinant was of finite order and an exact solution was obtainable. However, since the determinant of the linear system involves only $\chi_+(w)$, $\chi_-(w)$ and $\epsilon(w)$, and hence depends only upon $K(w)$, and, moreover, is non-singular except at eigenvalues for some forms of $K(w)$, it is a reasonable presumption that the determinant has no special degeneracy and that only exceptional forms for $K(w)$ will lead to non-uniqueness or non-convergence.

Finally an estimate of the accuracy of the approximation for $f_+(w)$ and $f_-(w)$ may be obtained by estimating the order of magnitude of the terms $R(w, \alpha')$ and $S(w, \beta')$. Using the easily proved results that $| f_+(w)e^{-iaw} | \to 0$ as Im $w \to \infty$ and $| f_-(w) | \to 0$ as Im $w \to -\infty$, we find that

$$| R(w,\alpha') | < Ke^{-a\alpha'}$$

and

$$| S(w,\beta') | < Me^{-a\beta'}$$

where K and M are constants. Thus the accuracy of the approximation improves exponentially with increasing α' and β'.

The corresponding approximation for $f_a(w)$ is determined by substituting (8) and (10) into (4), after neglecting $R(w, \alpha')$ and $S(w, \beta')$.

5. Approximation for $f(x)$. From the Fourier inversion formula and from (4), we we find that, for $0 \leq x \leq a$, $f(x)$ is given by

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-i\beta-\infty}^{-i\beta+\infty} dw' e^{-ixw'} f_a(w') \\ &= \frac{1}{2\pi}\int_{-i\beta-\infty}^{-i\beta+\infty} dw' e^{-ixw'}\left[\frac{e^{iaw'}}{\epsilon(w')\chi_-(w')}\left(\frac{f_+(w')\chi_+(w')e^{-iaw'}}{\epsilon(-w')}\right)\right. \\ &\quad \left. + \frac{\chi_+(w')}{\epsilon(-w')}\left(\frac{f_-(w')}{\epsilon(w')\chi_-(w')}\right) - \frac{\chi_+(w')}{\chi_-(w')}\frac{g_+(w') + g_-(w') + g_a(w')}{\epsilon(w')\epsilon(-w')}\right], \end{aligned} \tag{13}$$

where $-\infty < \beta < \infty$. For our purposes, we restrict β so that the contour in this equation lies below the zeros of $\epsilon(w)$ and $\chi_-(w)$, below the essential singularities, if any, of $\chi_-(w)$ and below the singularities of $g_-(w)$, but above the zeros of $\epsilon(-w)$ and poles of $\chi_+(w)$, above the singularities of $g_+(w)$, and above any essential singularities of $\chi_+(w)$. [For convenience it is assumed that $w = 0$ is not a root of $\epsilon(w)$]. Now in the first integral on the right-hand side of (13) we allow $-\beta$ to increase to, say, α'; in the second integral we allow $-\beta$ to decrease to, say, $-\beta'$. This procedure leads to the following expression for $f(x)$:

$$\begin{aligned} f(x) &= i\sum \text{Res.}\left[\frac{e^{i(a-x)w'}}{\epsilon(w')\chi_-(w')}\left(\frac{f_+(w')\chi_+(w')e^{-iaw'}}{\epsilon(-w')}\right)\right] + I(x,\alpha') \\ &\quad - i\sum \text{Res.}\left[\frac{e^{-ixw'}\chi_+(w')}{\epsilon(-w')}\left(\frac{f_-(w')}{\epsilon(w')\chi_-(w')}\right)\right] + J(x,\beta') \\ &\quad - \frac{1}{2\pi}\int_{-i\beta-\infty}^{-i\beta+\infty} dw' e^{-ixw'}\frac{g_+(w') + g_-(w') + g_a(w')}{\epsilon(w')\epsilon(-w')\chi_-(w')}\chi_+(w'), \end{aligned} \tag{14}$$

where

$$I(x,\alpha') = \frac{1}{2\pi}\int_{i\alpha'-\infty}^{i\alpha'+\infty} dw' \frac{e^{i(a-x)w'}}{\epsilon(w')\chi_-(w')}\left(\frac{f_+(w')\chi_+(w')}{\epsilon(-w')}e^{-iaw'}\right),$$

$$J(x,\beta') = \frac{1}{2\pi}\int_{-i\beta'-\infty}^{-i\beta'+\infty} dw' e^{-ixw'}\frac{\chi_+(w')}{\epsilon(-w')}\left(\frac{f_-(w')}{\epsilon(w')\chi_-(w')}\right),$$

and $0 \leq x \leq a$. The summations in (14) are over the residues at the poles crossed by the contours as $-\beta$ is increased to α' in the first integral of (13) and is decreased to $-\beta'$ in the second integral.

The final approximation to $f(x)$ is obtained by neglecting $I(x, \alpha')$ and $J(x, \beta')$ in (14) and by using the expressions (11) and (12), neglecting $R(w, \alpha')$ and $S(w, \beta')$, for $f_+(w)$ and $f_-(w)$. A straightforward calculation shows that the error resulting from neglect of $I(x, \alpha')$ and $J(x, \beta')$ is

$$| I(x,\alpha') | < Ae^{-(a-x)\alpha'}$$

and

$$| J(x,\beta') | < Be^{-\beta' x},$$

where A and B are constants. Thus for large α' and β', the error is small except in the neighborhood of $x = 0$ and $x = a$, which for increasing a becomes less and less important.

6. Eigenvalues of (1). In the case that $g(x) = 0$ in (1), we have an eigenvalue problem. Then (11) and (12) become a system of linear homogeneous equations for the determination of the values of $f_+(w)$ and $f_-(w)$ at the poles of the integrands of (8) and (10). The values of λ admitting of solutions of (11) and (12) under these circumstances will clearly then constitute approximate eigenvalues for (1).

7. Example:

$$f(x) = e^{-\alpha|x|} + \lambda \int_0^a dy\, e^{-|x-y|} f(y).$$

To illustrate the application of the present method, the above simple integral equation will be treated. This equation admits of an exact solution since the transform of the kernel is a rational fraction and, consequently, the series expansions in (11), (12) and (14) terminate.

From the definitions (i)–(vii) of Sec. 2 and property (iv) of Sec. 3, we find:

$$g_-(w) = \frac{1}{\alpha + iw},$$

$$g_+(w) = \frac{e^{a(iw-\alpha)}}{\alpha - iw},$$

$$g_a(w) = \frac{1 - e^{a(iw-\alpha)}}{\alpha - iw},$$

$$\lambda K(w) - 1 = \frac{2\lambda - 1 - w^2}{w^2 + 1},$$

$$\epsilon(w) = w - (2\lambda - 1)^{1/2},$$

$$\chi_+(w) = w + i,$$

$$\chi_-(w) = \frac{1}{w - i}.$$

Substitution of these quantities into (11) and (12) and evaluation of the residues leads to the result

$$f_-(w) = \frac{1}{\alpha + iw} + \left\{ \frac{(2\lambda - 1)^{1/2} - i}{\alpha - i(2\lambda - 1)^{1/2}} + [(2\lambda - 1)^{1/2} - i] f_+([2\lambda - 1]^{1/2}) \right\} \frac{1}{w - i},$$

$$f_+(w) = \frac{e^{a(iw-\alpha)}}{\alpha - iw} + \Bigg\{ [-(2\lambda - 1)^{1/2} + i] \cdot \left(\frac{1}{\alpha - i(2\lambda - 1)^{1/2}} + \frac{1 - \exp\{-a[\alpha + i(2\lambda - 1)^{1/2}]\}}{\alpha + i(2\lambda - 1)^{1/2}} \right) \cdot \exp[ia(2\lambda - 1)^{1/2}] + [(2\lambda - 1)^{1/2} - i] f_-(-[2\lambda - 1]^{1/2}) \cdot \exp[ia(2\lambda - 1)^{1/2}] \Bigg\} \frac{e^{iaw}}{w + i}.$$

Forming from these equations the linear system for evaluating $f_+([2\lambda - 1]^{1/2})$ and $f_-(-[2\lambda - 1]^{1/2})$ and carrying out the calculation gives

$$f_+([2\lambda - 1]^{1/2}) = \frac{1}{D}\frac{i(2\lambda - 1)^{1/2}}{\lambda(\alpha^2 + 2\lambda - 1)}\left\{\frac{1 + \alpha}{i(2\lambda - 1)^{1/2} - 1}\exp\left[ia(2\lambda - 1)^{1/2}\right] + \frac{1 - \alpha}{i(2\lambda - 1)^{1/2} + 1}e^{-a\alpha}\right\},$$

$$f_-(-[2\lambda - 1]^{1/2}) = \frac{1}{D}\frac{1}{\lambda(\alpha^2 + 2\lambda - 1)}\left\{\alpha\frac{i(2\lambda - 1)^{1/2} + 1}{i(2\lambda - 1)^{1/2} - 1}\exp\left[ia(2\lambda - 1)^{1/2}\right] + \frac{\alpha + i(2\lambda - 1)^{1/2}}{i(2\lambda - 1)^{1/2} + 1}\exp\left[-ia(2\lambda - 1)^{1/2}\right] + \frac{i(2\lambda - 1)^{1/2}(1 - \alpha)}{i(2\lambda - 1)^{1/2} - 1}e^{-a\alpha}\right\},$$

where the determinant, D, of the linear system is given by

$$D = \frac{\exp\left[-ia(2\lambda - 1)^{1/2}\right]}{(i[2\lambda - 1]^{1/2} + 1)^2} - \frac{\exp\left[ia(2\lambda - 1)^{1/2}\right]}{(i[2\lambda - 1]^{1/2} - 1)^2}.$$

The values of λ for which $D = 0$ are the eigenvalues. For the present case, $\lambda = 1/2$ is the smallest eigenvalue.

For $0 \leq x \leq a$, evaluation of the residues in (14) yields

$$f(x) = -\frac{i\lambda}{(2\lambda - 1)^{1/2}}f_+([2\lambda - 1]^{1/2})\exp\left[-ix(2\lambda - 1)^{1/2}\right] - \frac{i\lambda}{(2\lambda - 1)^{1/2}}\left\{f_-(-[2\lambda - 1]^{1/2}) - \frac{2\alpha}{\alpha^2 + 2\lambda - 1}\right\}\exp\left[ix(2\lambda - 1)^{1/2}\right] + \frac{\alpha^2 - 1}{\alpha^2 + 2\lambda - 1}e^{-\alpha x}.$$

If $D \neq 0$, the above expressions for $f_+([2\lambda - 1]^{1/2})$ and $f_-(-[2\lambda - 1]^{1/2})$ may be substituted into this equation for $f(x)$, which after rearranging gives the final result

$$f(x) = \left\{\frac{1 + \alpha}{i(2\lambda - 1)^{1/2} - 1}\exp\left[ia(2\lambda - 1)^{1/2}\right] + \frac{1 - \alpha}{i(2\lambda - 1)^{1/2} + 1}e^{-a\alpha}\right\}\cdot\frac{\exp\left[-ix(2\lambda - 1)^{1/2}\right]}{D(\alpha^2 + 2\lambda - 1)} + \left\{\frac{1 + \alpha}{i(2\lambda - 1)^{1/2} + 1}\exp\left[-ia(2\lambda - 1)^{1/2}\right] + \frac{1 - \alpha}{i(2\lambda - 1)^{1/2} - 1}e^{-a\alpha}\right\}\frac{\exp\left[ix(2\lambda - 1)^{1/2}\right]}{D(\alpha^2 + 2\lambda - 1)} + \frac{\alpha^2 - 1}{\alpha^2 + 2\lambda - 1}e^{-\alpha x}.$$

This simple example is typical of the application of the present method to equations for which $K(w)$ is a rational fraction. For other functions $K(w)$, the procedure is completely analogous, but has the same difficulty as with the usual Wiener-Hopf method that the functions $\chi_+(w)$ and $\chi_-(w)$ are in general not evaluable analytically but must be treated either numerically or approximately.

8. Example:

$$f(x) = \frac{\lambda}{(2\pi)^{1/2}}\int_0^a dy\,\exp\left[-\tfrac{1}{2}(x - y)^2\right]f(y).$$

For the case of the Gaussian kernel, the present method has been applied to the determination of the lowest eigenvalue. Previous consideration [3] of this problem has led to the calculation of upper and lower bounds for this eigenvalue. The present calculation, while illustrating the above method and indicating its accuracy and limitations, provides an extension of the earlier work on this problem by obtaining an accurate estimate of the eigenvalue in the region where the bounding values differ significantly.

Taking the transform of $k(x) = 1/(2\pi)^{1/2} \exp(-x^2/2)$, we find

$$\lambda K(w) - 1 = \lambda \exp(-\tfrac{1}{2}w^2) - 1.$$

The roots of this expression occur at

$$w_n = \pm [2 \ln \lambda + 4\pi n i]^{1/2},$$

where n is a positive or negative integer or zero and it is known [3] that $\lambda \geq 1$. For the approximation described below, we limit λ so that only the two roots with $n = 0$ lie within the strip $|\operatorname{Im} w| < 1$. This requires that $\lambda \lesssim 20$. With this restriction, we may write

$$[\lambda K(w) - 1] = \frac{\chi_-(w)}{\chi_+(w)} [-w - (2 \ln \lambda)^{1/2}][w - (2 \ln \lambda)^{1/2}].$$

From ([1], p. 339) it follows that

$$\chi_-(w) = \frac{\exp[-\tau_-(w)]}{w - i}$$

and

$$\chi_+(w) = -\frac{1}{\chi_-(-w)},$$

where

$$\tau_-(w) = \frac{1}{2\pi i} \int_{i\alpha-\infty}^{i\alpha+\infty} \frac{dw'}{w' - w} \ln \left\{ [1 - \lambda \exp(-\tfrac{1}{2}w'^2)] \frac{w'^2 + 1}{w'^2 - 2 \ln \lambda} \right\}$$

and the contour lies in the strip $0 < \operatorname{Im} w < 1$.

From (11) and (12), it is found that

$$\frac{f_-(w)}{\epsilon(w)\chi_-(w)} = \frac{f_+([2 \ln \lambda]^{1/2})}{\chi_-([2 \ln \lambda]^{1/2})} \frac{1}{(2 \ln \lambda)^{1/2} - w},$$

$$f_+(w) \frac{\chi_+(w)}{\epsilon_-(w)} e^{-iaw} = \frac{f_-(-[2 \ln \lambda]^{1/2})\chi_+(-[2 \ln \lambda]^{1/2}) \exp[ia(2 \ln \lambda)^{1/2}]}{(2 \ln \lambda)^{1/2} + w},$$

where residues at the roots w_n, $n \neq 0$, of $\lambda K(w) - 1$ have been neglected. These terms, as argued at the end of Sec. 4, are of the order of $\exp[-a \mid \operatorname{Im}(2 \ln \lambda \pm 4\pi i)^{1/2}]$. For $a \sim 1$, it may be shown from [3] that $\lambda \sim 2.5$ so that $\mid \operatorname{Im}(2 \ln \lambda \pm 4\pi i)^{1/2} \mid \sim 2.3$. The error is then of order $e^{-2.3} \sim 1/10$ and decreases rapidly as a increases.

The linear system for determining $f_+([2 \ln \lambda]^{1/2})$ and $f_-(-[2 \ln \lambda]^{1/2})$ may now be written

$$\frac{f_-(-[2 \ln \lambda]^{1/2})}{\chi_-(-[2 \ln \lambda]^{1/2})} - \frac{f_+([2 \ln \lambda]^{1/2})}{\chi_-([2 \ln \lambda]^{1/2})} = 0,$$

$$f_-(-[2 \ln \lambda]^{1/2})\chi_+(-[2 \ln \lambda]^{1/2}) \exp[ia(2 \ln \lambda)^{1/2}] + f_+([2 \ln \lambda]^{1/2})\chi_+([2 \ln \lambda]^{1/2}) \exp[-ia(2 \ln \lambda)^{1/2}] = 0.$$

The eigenvalues are determined by the solubility of this linear system which leads to the vanishing of the determinant of the system, namely,

$$\frac{\chi_+([2 \ln \lambda]^{1/2})}{\chi_-(-[2 \ln \lambda]^{1/2})} \exp [-ia(2 \ln \lambda)^{1/2}] - \frac{\chi_+(-[2 \ln \lambda]^{1/2})}{\chi_-([2 \ln \lambda]^{1/2})} \exp [ia(2 \ln \lambda)^{1/2}] = 0.$$

From the definitions of $\chi_+(w)$ and $\chi_-(w)$, we may re-express this equation as

$$\exp [i2a(2 \ln \lambda)^{1/2}] = \left(\frac{(2 \ln \lambda)^{1/2} + i}{(2 \ln \lambda)^{1/2} - i}\right)^2 \exp [-2\{\tau_-([2 \ln \lambda]^{1/2}) - \tau_-(-[2 \ln \lambda]^{1/2})\}].$$

As a further simplification the expression for $\tau_-([2 \ln \lambda]^{1/2})$ may be written

$$\tau_-([2 \ln \lambda]^{1/2}) = -\tfrac{1}{2} \ln \frac{2 \ln \lambda + 1}{2} + \frac{1}{2\pi i} J([2 \ln \lambda]^{1/2}),$$

where

$$J([2 \ln \lambda]^{1/2}) = P \int_{-\infty}^{\infty} \ln \left\{[1 - \lambda \exp (-\tfrac{1}{2}x^2)] \frac{x^2 + 1}{x^2 - 2 \ln \lambda}\right\} \frac{dx}{x - (2 \ln \lambda)^{1/2}}$$

and the integration is along the real axis, P standing for "principal value." Then the above expression becomes

$$\exp [ia(2 \ln \lambda)^{1/2}] = \left(\frac{(2 \ln \lambda)^{1/2} + i}{(2 \ln \lambda)^{1/2} - i}\right)^2 \exp \frac{-4J([2 \ln \lambda]^{1/2})}{2\pi i}.$$

A simple argument shows that this yields

$$a = \frac{2 \tan^{-1} \dfrac{1}{(2 \ln \lambda)^{1/2}} + \dfrac{1}{\pi} J([2 \ln \lambda]^{1/2}) + n\pi}{(2 \ln \lambda)^{1/2}}$$

where n is a positive or negative integer or zero. It may be shown that as $\lambda \to 1$, $J([2 \ln \lambda]^{1/2}) \to 0$ as $2 \ln \lambda$. Thus, since for the lowest eigenvalue $a \to \infty$ as $\lambda \to 1$, it is easily seen that $n = 0$. The larger eigenvalues correspond to $n > 0$.

An evaluation of $J([2 \ln \lambda]^{1/2})$ was carried out numerically so as to exhibit the values of λ as a function of a. The results are contained in the following table.

a	λ	I_L	I_U
6.03	1.10	1.00	1.15
4.03	1.20	1.05	1.25
2.65	1.40	1.23	1.43
2.05	1.60	1.44	1.62
1.70	1.80	1.65	1.81
1.46	2.00	1.87	2.01
1.10	2.50	2.39	2.50
.891	3.00	2.91	3.00
.747	3.50	3.43	3.51
.641	4.00	3.98	4.04
.559	4.50	4.54	4.60
.492	5.00	5.14	5.19
.390	6.00	6.47	6.51
.253	8.00	9.93	9.93

The quantities I_L and I_U are

$$I_L = \left\{ \frac{2}{(2\pi)^{1/2}} \int_0^{a/2} \exp(-\tfrac{1}{2}x^2)\, dx \right\}^{-1},$$

$$I_U = \left\{ \frac{2}{(2\pi)^{1/2}} \int_0^{a} \exp(-\tfrac{1}{2}x^2)\, dx - \frac{2}{a(2\pi)^{1/2}} [1 - \exp(-\tfrac{1}{2}a^2)] \right\}^{-1},$$

and correspond to the lower and upper bounds of the eigenvalue deduced in [3]. It is observed that to three figures the eigenvalue obtained above lies within the bounds I_L and I_U for $a \gtrsim 0.6$ which is consistent with the earlier error estimate. For smaller values of a the present approximation begins to break down as expected.

References

1. E. C. Titchmarsh, *Fourier integrals*, Oxford Press, 1937
2. R. Paley and N. Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc. Coll. Publ., 1934
3. R. Bellman and R. Latter, Proc. Amer. Math. Soc. **3**, 884 (1952)

BOOK REVIEWS

(*Continued from p. 20*)

Gas dynamics. By Klaus Oswatitsch. English version by Gustav Kuerti. Academic Press, Inc., New York, 1956. xv + 610 pp. $12.00.

The rapid development of supersonic aerodynamics during the past twenty years has created a need for authoritative text books. Since most people working in this field have been engaged very fully on highly specialized problems and since so many and varied contributions have been made to the subject it was inevitable that the first leading works of reference should have been prepared by teams of writers rather than by individuals. In the circumstances it was unavoidable that these works, excellent though they are, should suffer from a non-uniformity in style and, owing to difficulties in liaison, should have been partially outdated at the time of their publication.

Dr. Oswatitsch has therefore performed a great service in preparing the present volume single handed. He provides us with the opportunity of covering the complete range of Gas Dynamics in a style which is even, clear and easily followed throughout. The work is of considerable scope and covers almost every important aspect of the subject. While the treatment is thorough and rigorous the emphasis is on explaining the many interesting phenomena occurring in Gas Dynamics, without introducing complications due to Mathematical details or laboring particular practical applications. As the author himself states this is not an Engineering handbook, nor, it might be added, is it a Mathematical treatise. The aim is clearly to give an understanding of the physical character of the subject and this is certainly accomplished.

The book follows a logical order. Thermodynamic principles and one dimensional models of gas flows precede chapters on general equations and theorems. Three chapters follow on steady subsonic, steady supersonic and steady transonic flow respectively. This last chapter is of especial interest since it not only contains a much tidier account of recent work in the United States than has appeared elsewhere but also present the author's own approach to this subject with a clarity which the reviewer missed in earlier versions. After a fairly complete chapter on linearized theory the book concludes with a chapter on viscous effects and a chapter on experimental methods.

Dr. Kuerti is to be congratulated on his English translation of the book. Unlike some earlier English versions of German text books, which were often harder to follow than in the original, the present work reads as though it has been prepared in English. This is evidently the result of the translator taking the trouble to understand the whole contents and of his close collaboration with the author.

The book is very well set out; it is liberally illustrated with diagrams and remarkably clear photographs and the list of references is excellent both in its completeness and in being up to date. It would serve as a first rate text book in a graduate course but is also readily understood by the student working without supervision. It is strongly recommended to all those engaged in work on high speed aerodynamics.

M. Holt

Abacs or nomograms. An introduction to their theory and construction, illustrated by examples from engineering and physics. By A. Giet. Translated and revised by J. W. Head and H. D. Phippen. Iliffe & Sons, Ltd., London; Philosophical Library, New York, 1956. $10.00.

This book was originally published by Dunod in 1954 under the title "Abaques ou nomogrammes". The scope is indicated by the following chapter headings: Relations between two variables—Cartesian abacs—Alignment charts—Alignment charts not based on parallel coordinates—Relations between n variables. The mathematical treatment is elementary, avoiding even the use of determinants, but all important types of nomogram are thoroughly discussed and illustrated by numerous examples.

W. Prager

(*Continued on p. 46*)

ON THE HEAT TRANSFER TO CONSTANT-PROPERTY LAMINAR BOUNDARY LAYER WITH POWER-FUNCTION FREE-STREAM VELOCITY AND WALL-TEMPERATURE DISTRIBUTIONS*

BY

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Summary. The heat transfer to constant-property laminar boundary layer with power-function variations of free stream velocity ($u_1 = cx^m$) and of temperature difference between wall and free stream ($T_0 - T_1 = bx^n$) is studied by means of an improved version of the WKB method developed by the author. It is found that the local heat-transfer coefficient h can be approximately given in the form

$$\frac{hx/k}{(u_1x/\nu)^{1/2}} = \frac{1}{(2-\beta)^{1/2}}\left[\frac{\Gamma(2/3)}{3^{2/3}\Gamma(4/3)}\{\tfrac{1}{2} + n(2-\beta)\}^{1/3}(\sigma\alpha)^{1/3} - \frac{\beta}{10\alpha}\right],$$

where $\beta = 2m/(m+1)$, α is the non-dimensional velocity gradient at the wall (usually expressed as $\alpha = f''(0)$), σ is the Prandtl number, k is the thermal conductivity, and ν is the kinematic viscosity.

As examples of application of the formula two cases of special interest, i.e. the Blasius flow and stagnation flow are considered for several values of n, and the results are compared with the purely numerical computations by Levy and by Chapman and Rubesin, as well as with the analytical approximate formulas obtained by Schuh, by Fettis and by Lighthill. It appears that the above formula is the most accurate among those hitherto proposed.

1. Introduction. The calculation of the heat transfer to the laminar boundary layer with arbitrary distributions of main flow velocity and wall temperature is very difficult even for the constant-property flow, because the problem is governed by a system of non-linear partial differential equations. But, as was shown by Fage and Falkner [1] the system of equations can be reduced to that of ordinary differential equations if both the main-stream velocity and the wall temperature are assumed to vary as power functions of distance from the start of the boundary layer, namely $u_1 = cx^m$, $T_0 - T_1 = bx^n$. In fact, the momentum equation has been exactly solved by Hartree [2], and it remains only to solve the energy equation with variable coefficients given in terms of the Hartree solution. The exact treatment of the latter problem is naturally only possible by numerical means just as in the case of the momentum equation. Thus, Chapman and Rubesin [3] obtained the numerical solution for several values of n for the case of the Blasius flow ($m = 0$) by use of the Runge-Kutta integration method, assuming the Prandtl number σ to be 0.72. Later, Levy [4] carried out extensive numerical computations for a number of combinations of m, n, and σ by reducing the differential equation to finite difference form and employing the IBM machine. From his numerical results, Levy proposed an empirical formula for the local heat-transfer coefficient.

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In view of the considerable amount of labor involved in the numerical computations with three parameters m, n, and σ, Schuh [5] attempted to obtain the asymptotic solution of the energy equation when either n or σ is large so that the thermal boundary layer is very thin compared with the velocity boundary layer. Making further assumption that $\sigma/n \to 0$, he succeeded in obtaining the solution in a closed form using the Bessel function of order 1/3 and hence a very compact formula for the local heat-transfer coefficient. Also he showed that the formula gives numerical results in good agreement with Levy's numerical computations. Dropping the second assumption $\sigma/n \to 0$, Punnis [6] and Fettis [7] succeeded in finding the solution in a closed form in terms of the confluent hypergeometric functions. Especially the latter author derived an explicit formula for the local heat-transfer coefficient, but he did not make any numerical discussion of the formula obtained.

In a paper which probably escaped the attention of the above-mentioned three authors, Lighthill [8] had developed a method of approximate calculation of heat transfer to the laminar boundary layer with arbitrary distributions of wall temperature and main-stream velocity by making the simplifying assumption that the velocity profile in the boundary layer can be approximated by its tangent at the wall. As an example, the method was applied to the Blasius flow and the result was found to be in good agreement with Chapman-Rubesin's numerical calculation.

On the same assumption as made by Lighthill, Tamaki [9] independently developed an approximate method of calculating thermal boundary layers, especially for the case of the Blasius flow; his formula for the local heat-transfer coefficient is essentially the same as Lighthill's for this special case, but he treats a number of interesting cases using his formula and also extends the method so as to cover even the turbulent boundary layer. Further, he carried out numerical computations for our problem for the cases of $\beta = 2m/(m+1) = 1, 0.5, 0, -0.14, -0.1988$ and the range of n from -1.0 to 1.0 with $\sigma = 0.7$.

In the present paper, the basic energy equation is solved by use of the improved version of the WKB method proposed by the author [10, 11, 12] which has been shown by experience to give accurate results for various problems governed by the linear ordinary differential equation of the second order. The local heat-transfer coefficient is given in terms of algebraic functions of the three parameters m, n, and σ. Although the full expression obtained is somewhat complicated, it can be approximated by a rather simple expression. Applying the formulas, both full and simplified, to the cases of the Blasius flow ($m = 0$) and the stagnation flow ($m = 1$), it is found that they are in very good agreement with Levy's numerical computations.

2. Basic equations. For the laminar boundary layer on a body placed in an incompressible constant-property fluid, the continuity, momentum, and energy equations can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{dp}{dx} + \mu\frac{\partial^2 u}{\partial y^2}, \tag{2.2}$$

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2}, \tag{2.3}$$

where x and y are measured parallel and perpendicular to the surface of the body, u and v are velocity components, p is the pressure, T is the temperature, ρ is the density, C_p is the specific heat at constant pressure, k is the thermal conductivity, and μ is the viscosity. The boundary conditions are

$$T = T_0(x), \qquad u = v = 0 \quad \text{for} \quad y = 0 \tag{2.4}$$

$$T = T_1, \qquad u \to u_1(x) \quad \text{as} \quad y \to \infty \tag{2.5}$$

where $T_0(x)$ is the surface temperature and T_1 and $u_1(x)$ are respectively the temperature and velocity of the main stream.

In the following, it will be assumed that

$$T_0 - T_1 = bx^n, \tag{2.6}$$

$$u_1 = cx^m, \tag{2.7}$$

where b, c, m, and n are constants. In this case it is known that the system of partial differential equations (2.1), (2.2), and (2.3) can be reduced to a pair of ordinary differential equations. Thus, introducing the variable

$$\eta = \left(\frac{1 + m}{2}\frac{u_1}{\nu x}\right)^{1/2} y \tag{2.8}$$

and assuming that the stream function ψ defined by

$$u = \partial\psi/\partial y, \qquad v = -\partial\psi/\partial x \tag{2.9}$$

and the temperature T can be expressed in the forms

$$\psi = \left(\frac{2}{1 + m}\nu x u_1\right)^{1/2} f(\eta), \tag{2.10}$$

$$\frac{T - T_1}{T_0 - T_1} = \theta(\eta), \tag{2.11}$$

Eqs. (2.1), (2.2), and (2.3) are reduced to

$$f''' + ff'' + \beta(1 - f'^2) = 0, \tag{2.12}$$

$$\theta'' + \sigma f\theta' - \sigma n(2 - \beta)f'\theta = 0, \tag{2.13}$$

where $\beta = 2m/(m + 1)$ and $\sigma = \mu C_p/k$ is the Prandtl number. The boundary conditions are

$$f = f' = 0, \qquad \theta = 1 \quad \text{for} \quad \eta = 0 \tag{2.14}$$

and

$$f' \to 1, \qquad \theta \to 0 \quad \text{as} \quad \eta \to \infty. \tag{2.15}$$

Equation (2.12) is well known as the Falkner-Skan equation and has been solved by Hartree [2] for several values of β. Hence Eq. (2.13) can be regarded as a linear differential equation of the second order with known variable coefficients.

3. The improved WKB method. For the integration of Eq. (2.13) it is convenient

to make use of a refined version of the WKB method as developed by the present author [10, 11, 12]. Thus, consider a second-order linear differential equation

$$\frac{d^2\Phi}{dx^2} + a^2P(x)\Phi = 0, \qquad a \to \infty, \tag{3.1}$$

where a is a parameter and $P(x)$ has the form

$$P(x) = a_1x(1 + b_1x + b_2x^2 + \cdots), \qquad a_1 \neq 0. \tag{3.2}$$

Then, it can be shown that the solution of Eq. (3.1) can be expressed as

$$\Phi = P^{-1/4} \exp(Q/4a^2)\{C_1 \exp[i(az - Q_1/2a)] + C_2 \exp[-i(az - Q_1/2a)]\} + 0(a^{-3}) \quad \text{for} \quad x \neq 0 \tag{3.3}$$

and

$$\Phi = \left(\frac{\pi a}{2}\right)^{1/2} P^{-1/4}z^{1/6}\zeta\{C_1i^{5/6}H_{1/3}^{(1)}(\omega) + C_2i^{-5/6}H_{1/3}^{(2)}(\omega)\} + 0(a^{-3}) \quad \text{for} \quad x = 0(a^{-2/3}), \tag{3.4}$$

where

$$z = \int_0^x P^{1/2}\,dx, \tag{3.5}$$

$$Q = P^{-1/4}\frac{d^2P^{1/4}}{dz^2} = -P^{-3/4}\frac{d^2P^{-1/4}}{dx^2} = -\left(\frac{5}{36}\frac{1}{z^2} + \lambda z^{-2/3} + \lambda_0 + \lambda_1 z^{2/3} + \cdots\right), \tag{3.6}$$

$$Q_1 = \int_0^z \left(Q + \frac{5}{36}\frac{1}{z^2}\right)dz + \frac{5}{36}\frac{1}{z}, \tag{3.7}$$

$$\omega = \kappa(\xi + \tfrac{1}{5}\lambda_1\kappa^{-2}\xi^2)^{3/2}, \tag{3.8}$$

$$\zeta = (\xi - \tfrac{1}{5}\lambda_1\kappa^{-2}\xi^2)^{1/2}, \tag{3.9}$$

$$\xi = z^{2/3} + \lambda\kappa^{-2}, \tag{3.10}$$

$$\kappa^2 = a^2 + \lambda_0 \tag{3.11}$$

and the constants λ, λ_0, λ_1 are given in terms of a_1, b_1, $\cdots$ as follows:

$$\left.\begin{aligned} \lambda &= -\frac{3}{35}\left(\frac{3}{2}a_1\right)^{-2/3}(5b_2 - 3b_1^2), \\ \lambda_0 &= -\frac{4}{75}\frac{1}{a_1}(25b_3 - 35b_1b_2 + 14b_1^3), \\ \lambda_1 &= -\frac{27}{26950}\left(\frac{3}{2}a_1\right)^{-4/3}(6125b_4 - 4350b_2^2 \\ &\quad - 9800b_1b_3 + 12080b_1^2b_2 - 3624b_1^4). \end{aligned}\right\} \tag{3.12}$$

In order to consider the behavior of Φ as $x \to 0$, it is convenient to use $J_{\mp 1/3}(\omega)$ instead of $H_{1/3}^{(1,2)}(\omega)$. Thus, we have

$$H_{1/3}^{(1)} = \frac{2}{3^{1/2}}\, i^{2/3}(i^{-1/3}J_{1/3} - i^{1/3}J_{-1/3}), \tag{3.13}$$

$$H_{1/3}^{(2)} = \frac{2}{3^{1/2}}\, i^{-2/3}(i^{1/3}J_{1/3} - i^{-1/3}J_{-1/3}) \tag{3.14}$$

and

$$P^{-1/4}z^{1/6}J_{1/3}(\omega) = \left(\frac{3}{2}a_1\right)^{-1/6} \frac{2^{-1/3}}{\Gamma(4/3)}\lambda\kappa^{-5/3}\left[1 - \left\{\frac{1}{5}b_1 - \frac{2}{3}\lambda^{-1}\left(\frac{3}{2}a_1\right)^{1/3}\kappa^2\right\}x + \cdots\right], \tag{3.15}$$

$$P^{-1/4}z^{1/6}\zeta J_{-1/3}(\omega) = \left(\frac{3}{2}a_1\right)^{-1/6} \frac{2^{1/3}}{\Gamma(2/3)}\kappa^{-1/3}\left[1 - \left\{\frac{1}{5}b_1 + \left(\frac{3}{4}\lambda^2 + \frac{2}{15}\lambda_1\right)\left(\frac{3}{2}a_1\right)^{1/3}\kappa^{-2}\right\}x + \cdots\right]. \tag{3.16}$$

4. Solution of the energy equation. In order to apply the above formulas, we have first to bring the basic equation (2.13) into the standard form (3.1). This can be done by putting

$$\theta = \exp\left[-\frac{1}{2}\sigma\int_0^\eta f\, d\eta\right]\cdot\Theta \tag{4.1}$$

so that Eq. (2.13) becomes

$$\Theta'' - P(\eta)\Theta = 0, \tag{4.2}$$

where

$$P(\eta) = \frac{1}{4}\sigma^2 f^2 + \sigma\left\{\frac{1}{2} + n(2-\beta)\right\}f'. \tag{4.3}$$

Now, Eq. (4.3) should be expressed in a form similar to Eq. (3.2). For this purpose, differentiating Eq. (2.12) with respect to η three times, putting $\eta = 0$ each time, and using the boundary condition (2.14), we have

$$f = f' = 0, \quad f'' = \alpha, \quad f''' = -\beta, \quad f^{\mathrm{IV}} = 0, \quad f^{\mathrm{V}} = (2-\beta)\alpha^2, \quad f^{\mathrm{VI}} = 2(2-3\beta)\beta\alpha \tag{4.4}$$

for $\eta = 0$. Here we have put $f''(0) = \alpha$, whose values have been calculated by Hartree [2] for several values of β. Hence

$$f = \frac{1}{2}\alpha\eta^2 - \frac{1}{6}\beta\eta^3 + \frac{1}{5!}(2\beta - 1)\alpha^2\eta^5 + \frac{1}{6!}2(2-3\beta)\beta\alpha\eta^6 + \cdots \tag{4.5}$$

Substituting Eq. (4.5) into Eq. (4.3) we have

$$P(\eta) = A\eta\left[1 - \frac{\beta}{2\alpha}\eta + \left\{\frac{2\beta-1}{24}\alpha + \frac{1}{16}\frac{\alpha^2\sigma^2}{A}\right\}\eta^3 + \left\{\frac{(2-3\beta)\beta}{60} - \frac{1}{24}\frac{\alpha\beta\sigma^2}{A}\right\}\eta^4 + \cdots\right], \tag{4.6}$$

where

$$A = \alpha\sigma\{\tfrac{1}{2} + n(2 - \beta)\}. \tag{4.7}$$

Comparison of Eqs. (3.1) and (3.2) with Eqs. (4.2) and (4.6) shows that

$$a = e^{i\pi/2}, \tag{4.8}$$

$$\left.\begin{aligned} &a_1 = A, \qquad b_1 = -\frac{\beta}{2\alpha}, \qquad b_2 = 0, \\ &b_3 = \frac{\sigma^2\alpha^2}{16A} - \frac{(1 - 2\beta)\alpha}{24}, \qquad b_4 = \frac{(2 - 3\beta)\beta}{60} - \frac{\sigma^2\alpha\beta}{24A} \end{aligned}\right\}. \tag{4.9}$$

The formulas (3.3) and (3.4) have originally been obtained on the assumption that the parameter a is very large. At first sight Eq. (4.8) might seem to conflict with this assumption. Since, however, the condition for the validity of the formulas is essentially that $|az|$ should be large, the condition is readily seen to be satisfied if A is large. This latter condition is satisfied if either the Prandtl number σ or the exponent n is large as was assumed by Schuh [5]. Furthermore, it should be mentioned that the formulas have been found to give accurate results in various cases of application even when a is not very large.

Let us assume $A > 0$. Then $P(\eta) > 0$ for $\eta > 0$ Consequently

$$z = \int_0^\eta P^{1/2}\, d\eta > 0 \tag{4.10}$$

so that

$$iaz = -z < 0.$$

Therefore, by the boundary condition (2.15), which can also be expressed as

$$\Theta \to 0 \quad \text{as} \quad z \to \infty$$

we have to take $C_2 = 0$ in Eq. (3.3). Thus

$$\Theta = C_1 P^{-1/4} \exp\left[-(z + \tfrac{1}{2}Q_1 + \tfrac{1}{4}Q)\right] \quad \text{for} \quad \eta \neq 0 \tag{4.11}$$

and

$$\Theta = C_1\left(\frac{\pi i}{2}\right)^{1/2} P^{-1/4} z^{1/6} \zeta i^{5/6} H_{1/3}^{(1)}(\omega) \quad \text{for} \quad \eta \doteqdot 0. \tag{4.12}$$

The constant C_1 can be determined by the boundary condition (2.14) as

$$C_1 = -(2/\pi)^{1/2}[H(0)]^{-1}. \tag{4.13}$$

Hence Eq. (4.12) becomes

$$\Theta = H(\eta)/H(0) \quad \text{for} \quad \eta \doteqdot 0, \tag{4.14}$$

where

$$H(\eta) = P^{-1/4} z^{1/6} \zeta i^{-2/3} H_{1/3}^{(1)}(\omega). \tag{4.15}$$

Now the heat flow rate at the wall is given by

$$q_0 = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = -k\,\frac{T_0 - T_1}{(2 - \beta)^{1/2}}\left(\frac{u_1}{\nu x}\right)^{1/2}\left(\frac{d\theta}{d\eta}\right)_{\eta=0}. \tag{4.16}$$

Therefore, if the local heat-transfer coefficient h is defined by

$$h = \frac{q_0}{T_0 - T_1}, \tag{4.17}$$

we have

$$\frac{hx/k}{(u_1 x/\nu)^{1/2}} = -(2 - \beta)^{-1/2}\left(\frac{d\theta}{d\eta}\right)_{\eta=0}. \tag{4.18}$$

But, by Eqs. (4.1) and (4.14)

$$\left(\frac{d\theta}{d\eta}\right)_{\eta=0} = \left(\frac{d\Theta}{d\eta}\right)_{\eta=0} = \frac{H'(0)}{H(0)}. \tag{4.19}$$

Now, combining Eqs. (3.6), (3.8), and (3.9), we have

$$H(0) = -\frac{2^{11/6}}{3^{2/3}} \frac{[(1 - \lambda_0)A]^{-1/6}}{\Gamma(2/3)} \left\{1 + \frac{2^{-2/3}\Gamma(2/3)}{\Gamma(4/3)} \lambda(1 - \lambda_0)^{-2/3}\right\}, \tag{4.20}$$

$$\frac{H'(0)}{H(0)} = -\frac{1}{5} b_1 - \left(\frac{2}{3}\right)^{2/3} [(1 - \lambda_0)A]^{1/3} \frac{1 - c[(9/8)\lambda^2 + \frac{1}{5}\lambda_1](1 - \lambda_0)^{-4/3}}{c + \lambda(1 - \lambda_0)^{-2/3}}, \tag{4.21}$$

where

$$c = \frac{2^{2/3}\Gamma(4/3)}{\Gamma(2/3)} = 1.04682 \tag{4.22}$$

and the constants b_1, λ, λ_0, λ_1 are given by Eqs. (3.12) and (4.9) as follows

$$\left.\begin{aligned} \lambda &= \frac{9}{35}\left(\frac{3}{2}A\right)^{-2/3} b_1^2, \\ \lambda_0 &= -\frac{4}{75} A^{-1}(25b_3 + 14b_1^3), \\ \lambda_1 &= -\frac{27}{26950}\left(\frac{3}{2}A\right)^{-4/3}(6125b_4 - 9800b_1 b_3 - 3624b_1^4), \end{aligned}\right\} \tag{4.23}$$

where

$$\left.\begin{aligned} b_1 &= -\frac{\beta}{2\alpha}, \qquad b_3 = -\frac{(1 - 2\beta)\alpha}{24} + \frac{\sigma^2\alpha^2}{16A}, \\ b_4 &= \frac{(2 - 3\beta)\beta}{60} - \frac{\sigma^2\alpha\beta}{24A}, \qquad A = \{\tfrac{1}{2} + n(2 - \beta)\}\sigma\alpha. \end{aligned}\right\} \tag{4.24}$$

If it is assumed that A is very large while b_1, b_3, and b_4 are $0(1)$ (this is true if n is very large so that $A = 0(n)$), Eqs. (4.23) give

$$\lambda = 0(A^{-2/3}), \qquad \lambda_0 = 0(A^{-1}), \qquad \lambda_1 = 0(A^{-4/3}).$$

Hence Eq. (4.21) can be approximated by

$$\frac{H'(0)}{H(0)} = \frac{\beta}{10\alpha} - \left(\frac{1}{3}\right)^{2/3} \frac{\Gamma(2/3)}{\Gamma(4/3)} A^{1/3} + 0(A^{-1/3}). \tag{4.25}$$

5. Comparison with previous investigations. The temperature distribution in the boundary layer is given by Eqs. (4.1), (4.11), (4.13), and (4.14) in the form

$$\theta(\eta) = -\left(\frac{2}{\pi}\right)^{1/2} \frac{P^{-1/4}}{H(0)} \exp\left[-\left(\tfrac{1}{2}\sigma \int_0^\eta f\, d\eta + z + \tfrac{1}{2}Q_1 + \tfrac{1}{4}Q\right)\right] \quad \text{for} \quad \eta \neq 0 \tag{5.1}$$

and

$$\theta(\eta) = \frac{H(\eta)}{H(0)} \exp\left[-\tfrac{1}{2}\sigma \int_0^\eta f\, d\eta\right] \quad \text{for} \quad \eta \doteqdot 0. \tag{5.2}$$

In the following we shall restrict ourselves to the heat-transfer coefficient h, which is given by

$$\frac{hx/k}{(u_1 x/\nu)^{1/2}} = -(2-\beta)^{-1/2} \frac{H'(0)}{H(0)}. \tag{5.3}$$

Here $H'(0)/H(0)$ is given by Eq. (4.21) or approximately by Eq. (4.25).

First let us compare our result with the results of previous investigators.

Schuh [5] considers that the thickness of the thermal boundary layer becomes small compared to the velocity boundary layer if either n or σ becomes large, and that it would then be sufficient to replace the velocity profile by its tangent at the wall, since, for calculating the temperature field, only that part of the velocity profile is important which lies within the thermal boundary layer. Thus he puts

$$df/d\eta = \alpha\eta \tag{5.4}$$

so that Eq. (2.13) reduces to

$$\theta'' + \tfrac{1}{2}\alpha\sigma\eta^2\theta' - \sigma n\alpha(2-\beta)\eta\theta = 0 \tag{5.5}$$

which, on putting

$$\zeta = n^{1/3}\eta \tag{5.6}$$

becomes

$$\frac{d^2\theta}{d\zeta^2} + \frac{\sigma}{n}\frac{\alpha}{2}\zeta^2 \frac{d\theta}{d\zeta} - \sigma\alpha(2-\beta)\zeta\theta = 0. \tag{5.7}$$

Further he assumes that

$$\sigma/n \to 0 \tag{5.8}$$

so that Eq. (5.7) simplifies itself to

$$d^2\theta/d\zeta^2 - \sigma\alpha(2-\beta)\zeta\theta = 0. \tag{5.9}$$

This can be integrated in terms of the Bessel function of order 1/3. The final result for the heat-transfer coefficient is

$$\frac{hx/k}{(u_1 x/\nu)^{1/2}} = \frac{\Gamma(2/3)}{3^{2/3}\Gamma(4/3)} \frac{(\sigma\alpha n)^{1/3}}{(2-\beta)^{1/6}}. \tag{5.10}$$

Next he tried to obtain an asymptotic solution of Eq. (5.7) which is valid for $\sigma \to \infty$ with no restriction on n. For this purpose, he introduced a new variable ξ defined by

$$\xi = (\sigma\alpha)^{1/3}\eta \tag{5.11}$$

to write Eq. (5.7) in the form

$$d^2\theta/d\xi^2 + \tfrac{1}{2}\xi^2\, d\theta/d\xi - \epsilon\xi\theta = 0 \tag{5.12}$$

with

$$\epsilon = n(2 - \beta). \tag{5.13}$$

However, he could not find any solution of Eq. (5.12) in a closed form and made no attempt to solve it numerically. As examples of application of the formula (5.10), he calculated the values of $(hx/k)(u_1x/\nu)^{-1/2}$ for the Blasius flow ($\beta = 0$) and for the stagnation flow ($\beta = 1$) for several values of n and compared the results with the purely numerical calculations by Chapman and Rubesin [3] and by Levy [4].

Later, Punnis [6] and Fettis [7] independently showed that Eq. (5.12) can be integrated in a closed form using confluent hypergeometric functions. Furthermore, the latter gave an explicit expression for the heat-transfer coefficient at the wall. His result for the temperature distribution function θ is expressed as

$$\theta = \frac{\Gamma(1 + 2\epsilon/3)}{\Gamma(1/3)}\,\xi^{-1}\exp(-\xi^{3/12})W_{-(1+2\epsilon/3),\,1/6}(\xi^3/6), \tag{5.14}$$

where $W_{k,m}(t)$ denotes the Whittaker function and ξ is defined by Eq. (5.11). The heat-transfer coefficient is given by the formula

$$\frac{hx/k}{(u_1x/\nu)^{1/2}} = -\frac{(\sigma\alpha/6)^{1/3}}{(2 - \beta)^{1/2}}\,\frac{\Gamma(1 + 2\epsilon/3)\Gamma(-1/3)}{\Gamma(2/3 + 2\epsilon/3)\Gamma(1/3)}. \tag{5.15}$$

Now it should be mentioned that Lighthill [8] had previously obtained a general formula which gives the heat transfer at the wall in terms of the skin friction for arbitrary main-stream velocity and wall-temperature distributions, based on the simplifying assumption adopted by Fage and Falkner [1]:

$$u = (1/\mu)\tau(x)y, \tag{5.16}$$

where $\tau(x)$ is the skin friction so that $\tau(x) = \mu(\partial u/\partial y)_{y=0}$. This assumption is of course the same as Eq. (5.4). Lighthill's formula can be written in our notation as follows.

$$q_0(x) = k\left(\frac{\sigma\rho}{9\mu^2}\right)^{1/3}\frac{[\tau(x)]^{1/2}}{\Gamma(4/3)}\int_0^x\left(\int_{x_1}^x [\tau(z)]^{1/2}\,dz\right)^{-1/3} dT_0(x_1). \tag{5.17}$$

Lighthill himself applied this formula to the case of the Blasius flow in order to test its accuracy.

Now the formula (5.17) will be applied to our problem defined by Eqs. (2.6) and (2.7). First we have, by Eqs. (2.8), (2.9), and (2.10),

$$\tau(x) = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu u_1 f''(0)\left(\frac{1 + m}{2}\,\frac{u_1}{\nu x}\right)^{1/2}. \tag{5.18}$$

Introducing Eqs. (5.18), (2.6), and (2.7) into Eq. (5.17) and making some calculation, we obtain

$$\frac{q_0(x)}{(T_0 - T_1)k(u_1/\nu x)^{1/2}} = -\frac{(\sigma\alpha/6)^{1/3}}{(2 - \beta)^{1/2}}\,\frac{\Gamma(1 + 2\epsilon/3)}{\Gamma(2/3 + 2\epsilon/3)}\,\frac{\Gamma(-1/3)}{\Gamma(1/3)} \tag{5.19}$$

which is nothing other than the formula (5.15). Therefore, it may be stated that Fettis' result is included in Lighthill's general formula. This is perhaps what should have been expected, because both results have been obtained under the same simplifying assumption about the velocity profile. Further it may be remarked that some idea of the accuracy of Lighthill's formula (5.17) may be obtained by comparison of the result (5.19) with our and other calculations.

Extensive numerical computations have been performed by Levy [4] by reducing Eq. (2.13) to finite difference form and employing an electronic IBM machine; the results cover a range of n from -2.5 to 4.0 and σ from 0.70 to 20.0 for four values of m(4, 1, 0, -0.0904). For the range of the calculations, it was found that the local heat-transfer coefficient can, with the exception of large negative n values, be expressed within 5 per cent as

$$\frac{hx/k}{(u_1x/\nu)^{1/2}} = B(m, n)\sigma^{\gamma}, \tag{5.20}$$

where the function $B(m, n)$ can be approximated by

$$B(m, n) = 0.57\left(\frac{2m}{m+1} + 0.205\right)^{0.104}\left(\frac{2n}{m+1} + 1\right)^{0.37+0.12m/(m+1)} \tag{5.21}$$

and the exponent γ of the Prandtl number varies as shown in the following table:

β	1.6	1.0	0	-0.199
γ	0.367	0.355	0.327	0.254

6. Blasius flow. For the case of the Blasius flow we have $\beta = 0$, so that Eqs. (4.24) and (4.23) give

$$b_1 = 0, \qquad b_3 = \frac{\sigma^2\alpha^2}{16A} - \frac{\alpha}{24}, \qquad b_4 = 0$$

$$\lambda = 0, \qquad \lambda_0 = -\frac{1}{12}\frac{\sigma^2\alpha^2}{A^2} + \frac{1}{18}\frac{\alpha}{A}, \qquad \lambda_1 = 0.$$

Hence Eqs. (5.3) and (4.21) become

$$\frac{hx/k}{(u_1x/\nu)^{1/2}} = \frac{1}{2^{1/2}}\frac{\Gamma(2/3)}{3^{2/3}\Gamma(4/3)}A^{1/3}\left(1 - \frac{1}{18}\frac{\alpha}{A} + \frac{1}{12}\frac{\sigma^2\alpha^2}{A^2}\right)^{1/3} \tag{6.1}$$

which can be approximated by

$$\frac{hx/k}{(u_1x/\nu)^{1/2}} = \frac{1}{2^{1/2}}\frac{\Gamma(2/3)}{3^{2/3}\Gamma(4/3)}A^{1/3}, \tag{6.2}$$

where

$$A = (\tfrac{1}{2} + 2n)\sigma\alpha, \qquad \alpha = 0.4696. \tag{6.3}$$

In this case, Schuh's formula (5.10) becomes

$$\frac{hx/k}{(u_1x/\nu)^{1/2}} = \frac{\Gamma(2/3)(n\sigma\alpha)^{1/3}}{2^{1/6}3^{2/3}\Gamma(4/3)} \tag{6.4}$$

and Fettis' formula (5.15) becomes

$$\frac{hx/k}{(u_1x/\nu)^{1/2}} = 2^{-1/2}6^{-1/3}\frac{\Gamma(2/3)\Gamma(1+4n/3)}{\Gamma(4/3)\Gamma(2/3+4n/3)}(\sigma\alpha)^{1/3} \tag{6.5}$$

which is essentially the same as the formula given by Lighthill. Table 1 shows the values of $(hx/k)(u_1x/\nu)^{-1/2}$ for several values of n as calculated by the various formulas. In Table 1 the first column shows the values calculated by Chapman and Rubesin by the use of the Runge-Kutta integration method. The Levy values were computed by the present author from Levy's values for $\sigma = 0.7$ by using Levy's empirical formula (5.20).

TABLE 1

Local heat-transfer coefficients for Blasius flow ($\beta = 0$). Prandtl number = 0.72.

n	Chapman-Rubesin	Levy	Lighthill Fettis	Schuh	Imai	Imai*
0	0.296	0.2901	0.304	0	0.3011	0.2850
1	0.489	0.4814	0.490	0.4524	0.4845	0.4874
2	0.597	0.5865	0.594	0.5700	0.5903	0.5929
3	0.684	0.6634	0.671	0.6525	0.6680	0.6702
4	0.744	0.7258	0.734	0.7182	0.7310	0.7329
5	0.799	–	0.787	0.7737	0.7847	0.7864
10	1.006	–	0.987	0.9747	0.9817	0.9829

The Lighthill values have been reproduced from his paper [8] and are the same as those given by Eq. (6.5). The "Imai" values are those calculated by use of the author's full expression (6.1), while the "Imai*" values are those calculated by the simplified expression (6.2). It will be seen that the values calculated by the author's formulas, both full and simplified, are in agreement with those computed by Levy within 1 per cent, except for the case of $n = 0$, where the difference is less than 2 per cent and 4 per cent, respectively, for the full and the simplified formula. By performing detailed examination of the accuracy of his numerical computations, Levy has estimated the accuracy of his results to be within 2 per cent. Therefore, our formula (6.1) and (6.2) may be considered to be perfectly satisfactory. Moreover, because of the simplicity, the simplified formula may be especially recommended for practical purposes.

Further it may be observed that Lighthill and Fettis' formula is better than Schuh's, which, however, is fairly good except for the case of $n = 0$. This rather high accuracy of the formulas of Schuh and of Lighthill and Fettis is naturally due to the fact that the simplifying assumption $u \propto y$ is very accurate for the Blasius flow.

7. Stagnation flow. Similar comparison is made for the case of the stagnation flow ($\beta = 1$), taking $\sigma = 1$ and 10.

Table 2 shows the case of $\sigma = 1$. Here the "Levy" values have been reproduced from his paper, while the "Levy*" values are those computed by the present author using the empirical formula (5.21). The "Fettis" and "Schuh" values were also computed by the author by use of Eqs. (5.10) and (5.15) respectively. The "Imai" and "Imai*" values are those calculated by the author's formulas (4.19) and (4.23) respectively.

It will be seen that both the Fettis and Schuh values are not in very good agreement with the Levy values. It is interesting to note that the Schuh values are better than the Fettis values except for small values of n, in spite of the fact that the former is an approxi-

TABLE 2

Local heat-transfer coefficients for stagnation flow ($\beta = 1$). Prandtl number = 1.

n	Levy	Levy*	Fettis	Schuh	Imai	Imai*
0	0.5603	0.581	0.661	0	0.584	0.539
0.25	0.6345	0.640	0.735	0.492	0.645	0.629
0.5	0.6979	0.692	0.799	0.620	0.705	0.701
1	0.8116	0.783	0.905	0.782	0.809	0.814
2	0.9647	0.932	1.065	0.985	0.971	0.980
3	1.089	1.055	1.189	1.127	1.096	1.106
4	1.192	1.161	1.292	1.241	1.200	1.209
5	–	1.256	1.382	1.337	1.289	1.299
10	–	1.630	1.693	1.684	1.622	1.630

mation to the latter. As has been remarked by Schuh, this is probably due to the fact that the following two influences cancel each other: (a) approximating the velocity profile by its tangent at the wall increases h for $\beta > 0$; (b) neglecting the middle term in Eq. (5.7) reduces h.

On the other hand, both the simplified and full formulas of the author give results in satisfactory agreement with the Levy values. It will be observed that the approximate "Imai*" values are even better than the "Levy*" values calculated by the rather complicated empirical formula.

Table 3 shows a similar comparison for the case of rather high Prandtl number of $\sigma = 10$. The situation is very similar to the above case of $\sigma = 1$. Here the superiority of the Schuh formula to the Fettis for $n > 1$ is rather more conspicuous. However, our simplified formula is seen to be in closest agreement with Levy's computation.

8. Conclusion. Summarizing the result of the above comparison it may be stated that our approximate formula

$$\frac{hx/k}{(u_1x/\nu)^{1/2}} = \frac{1}{(2-\beta)^{1/2}}\left[\left(\frac{1}{3}\right)^{2/3}\frac{\Gamma(2/3)}{\Gamma(4/3)}\{\tfrac{1}{2}+n(2-\beta)\}^{1/3}(\sigma\alpha)^{1/3}-\frac{\beta}{10\alpha}\right]$$

TABLE 3

Local heat-transfer coefficients for stagnation flow ($\beta = 1$). Prandtl number = 10.

n	Levy	Levy*	Fettis	Schuh	Imai*
0	1.286	1.316	1.424	0	1.255
0.25	1.451	1.449	1.584	1.061	1.449
0.5	1.590	1.567	1.721	1.337	1.603
1	1.818	1.773	1.949	1.684	1.847
2	2.159	2.111	2.295	2.122	2.204
3	2.418	2.389	2.562	2.429	2.476
4	2.630	2.629	2.784	2.673	2.699
5	–	2.844	2.976	2.880	2.891
10	–	3.691	3.649	3.628	3.606

is the best of the formulas hitherto proposed in view of the fact that it shows the effect of the parameters β (or m), n and σ in a simple functional form. Especially it will be interesting to note that the exponent of σ is invariably 1/3 and that the effect of the curvature of the velocity profile at the wall appears as an additive constant $-\beta/10\alpha$ in the formula.

References

1. A. Fage and V. M. Falkner, *Relation between heat transfer and surface friction for laminar flow*, A.R.C.R. & M., No. 1408, London, 1931
2. D. R. Hartree, *On an equation occurring in Falkner and Skan's approximate treatment of the equation of the boundary layer*, Proc. Cambridge Phil. Soc. **33**, 223-239 (1937)
3. D. R. Chapman and M. W. Rubesin, *Temperature and velocity profiles in the compressible laminar boundary layer with arbitrary distribution of surface temperature*, J. Aeronaut. Sci. **16**, No. 9, 547-565 (Sept. 1949)
4. S. Levy, *Heat transfer to constant-property laminar boundary-layer flows with power-function free-stream velocity and wall-temperature variation*, J. Aeronaut. Sci. **19**, No. 5, 341-348 (May 1952)
5. H. Schuh, *On asymptotic solutions for the heat transfer at varying wall temperatures in a laminar boundary layer with Hartree's velocity profiles*, J. Aeronaut. Sci. **20**, No. 2, 146-147 (Feb. 1953)
6. B. Punnis, *A remark on "On asymptotic solutions for the heat transfer at varying wall temperatures in a laminar boundary layer with Hartree's velocity profiles,"* J. Aeronaut. Sci. **20**, No. 7, 505 (July 1953)
7. H. E. Fettis, *On a differential equation occurring in the theory of heat flow in boundary layers with Hartree's velocity profiles*, J. Aeronaut. Sci. **21**, No. 2, 132-133 (Feb. 1954)
8. M. J. Lighthill, *Contributions to the theory of heat transfer through a laminar boundary layer*, Proc. Roy. Soc. London **A202**, No. 1070, 359-377 (1950)
9. H. Tamaki, *Studies of surface heat transfer based on the boundary layer theory*, (in Japanese), Rept. Inst. Ind. Sci., Univ. of Tokyo **1**, No. 8, 239-273 (March 1951)
10. I. Imai, *On a refinement of the WKB method*, Phys. Rev. **74**, 113 (July 1, 1948)
11. I. Imai, *Asymptotic solutions of ordinary linear differential equations of the second order*, Phys. Rev. **80**, 1112 (Dec. 15, 1950)
12. I. Imai, *A refinement of the WKB method and its application to the electromagnetic wave theory*, IRE Trans. **AP-4**, No. 3, 233-239 (July 1956); proceedings of the Symposium on Electromagnetic Wave Theory held June 20-25, 1955, at the University of Michigan

BOOK REVIEWS

(*Continued from p. 32*)

Elements of partial differential equations. By Ian N. Sneddon. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1957. ix + 327 pp. $7.50.

In the author's own words: "the aim of this book is to present the elements of the theory of partial differential equations in a form suitable for the use of students and research workers whose main interest in the subject lies in finding solutions of particular equations rather than in the general theory." An attempt has been made to touch upon subjects from the entire field of partial differential equations. There are six chapters in all. Chapter 1 is introductory in character and deals with ordinary differential equations in more than two independent variables. Of special interest here is the treatment of Pfaffian differential forms, Carathéodory's theorem on integrability, and its application to thermodynamics. Partial differential equations of the first order occupy Chapter 2. Among the subjects covered are Cauchy's problem for first order equations, linear and nonlinear equations, Cauchy's method of characteristics, Lagrange's and Charpit's methods and applications, for example to stochastic processes and to birth and death processes in bacteria. Chapter 3 is concerned with a preliminary discussion of second (and higher) order equations, preparatory to the last three chapters, which consider in more detail the three main types of second order equations. One finds a treatment of linear partial differential equations with constant coefficients, canonical forms for second order equations, characteristics of equations in two and three independent variables, Riemann's method of integration, and integral transforms. Chapter 4 deals with Laplace's equation, the prototype of the elliptic second order linear partial differential equations. The Dirichlet, Neumann and mixed boundary value problems are dealt with, Green's functions, Kelvin's inversion theorem, and allied topics. The typical hyperbolic equation, the wave equation, constitutes the subject matter of Chapter 5. One, two, and three space dimensions are considered; the Riemann-Volterra solution of the one dimensional equation, and Helmholtz', Weber's and Kirchhoff's general solutions in two and three dimensions. There is a brief introduction to Marcel Riesz' method of solution, based on Riesz' generalization of the Riemann-Liouville integral. The concluding Chapter 6 has the heat equation, the typical parabolic equation, as its topic. There is an appendix on systems of surfaces, meant to provide a brief outline of some of the properties of systems of surfaces used in Chapter 2. The use of this book as a text is facilitated by the presence of a large number of problems, together with the solutions of all odd numbered ones.

J. B. Diaz

The plastic methods of structural analysis. By B. G. Neal. John Wiley & Sons, Inc., New York, 1956. xi + 353 pp. $7.50.

"The fundamental defect in the orthodox elastic design technique can now be exposed. It has been seen that a simply supported beam carrying a uniformly distributed load . . . has a load factor against collapse of 1.75, whereas if a similarly loaded but fixed-ended beam were designed elastically to the same working stress its load factor against collapse would be 2.34." With these words midway through the introduction, Professor Neal embarks on a careful and detailed study of the status of beams and frames. His own work and the work of former colleagues at Cambridge and at Brown play a central role in the text along with reference to the many other workers in this country and abroad. The book is intended as a step by step guide through most of the complexities of the actual calculation of collapse loads, deflection estimates, effects of shear and axial force on limit moment, minimum weight design, and problems of shakedown.

Mathematics is kept to the barest minimum even in the proofs of the theorems which are relegated to the Appendices. Except for providing the necessary warnings about the possibility of buckling, Bauschinger effects, non-homogeneity of material and similar difficulties, the author is not deflected in his purpose. He demonstrates the power and the relative simplicity of plastic methods of analysis for frames encountered by the practicing engineer who should read all of the text and check his solution to the problems with the answers in the back of the book.

D. C. Drucker

(*Continued on p. 72*)

ASYMPTOTIC SOLUTIONS OF TOROIDAL SHELL PROBLEMS*

BY

R. A. CLARK

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1. Introduction. A few years ago a method of asymptotic integration was developed by E. Reissner and the author for a non-homogeneous differential equation involved in certain problems of the theory of thin elastic toroidal shells. Using this method solutions were found for the problem of bending a curved tube (considered as a sector of a shell of revolution) [4] and for the mathematically similar problem of a toroidal expansion joint subject to an axial force [2]. The main purpose of this paper is to indicate how the method used before can be refined and how the approximate solutions previously obtained for these problems can be generalized and extended.

It is shown, in particular, that the previous solutions are the leading terms of asymptotic expansions in inverse powers of a large parameter. Higher-order terms or corrections of two different types are obtained for toroidal shells of circular cross section and uniform thickness. Some of the results are found by specializing solutions derived first for toroidal shells of arbitrary cross section and possibly varying thickness, assuming only that the shell has a plane of symmetry perpendicular to the axis of revolution.

2. Basic equations for shells of revolution. The problem of determining rotationally-symmetric stress distributions in thin elastic shells of revolution may be reduced to that of solving two coupled second-order differential equations. Using the formulation and essentially the notation of E. Reissner [9] the differential equations for small deformations may be written in the form

$$\begin{aligned} X'' - \Theta_1 X + \mu\Phi Y &= F, \\ Y'' - \Theta_2 Y - \mu\Phi X &= G. \end{aligned} \tag{1}$$

Here primes denote differentiation with respect to an independent variable ξ which is the parameter in a representation of a meridian of the middle shell surface by equations of the form

$$r = r(\xi), \qquad z = z(\xi), \tag{2}$$

where r, θ, z are the cylindrical coordinates of a point on the middle surface. Representation (2) is assumed to be such that r and z have continuous derivatives up to at least third order and also such that the quantity

$$\alpha = [(r')^2 + (z')^2]^{1/2} \tag{3}$$

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never vanishes. The coefficients and loading terms in Eqs. (1) are given by

$$\Theta_1 = \left(\frac{r'}{r}\right)^2 - \nu\,\frac{(r'D/\alpha)'}{(rD/\alpha)} + \frac{1}{4}\left[\frac{(rD/\alpha)'}{(rD/\alpha)}\right]^2 + \frac{1}{2}\left[\frac{(rD/\alpha)'}{(rD/\alpha)}\right]',$$

$$\Theta_2 = \left(\frac{r'}{r}\right)^2 + \nu\,\frac{(r'/\alpha C)'}{(r/\alpha C)} + \frac{1}{4}\left[\frac{(r/\alpha C)'}{(r/\alpha C)}\right]^2 + \frac{1}{2}\left[\frac{(r/\alpha C)'}{(r/\alpha C)}\right]',$$

$$\mu\Phi = [12(1-\nu^2)]^{1/2}\,\frac{\alpha z'}{rh}, \qquad F = r'\left(\frac{\alpha}{rD}\right)^{1/2}(rV), \tag{4}$$

$$G = r'\left(\frac{\alpha C}{r}\right)^{1/2}k + \left(\frac{r}{\alpha C}\right)^{1/2}\left\{\left[\frac{r'z'}{r^2} + \nu\,\frac{(z'/\alpha C)'}{(r/\alpha C)}\right](rV) + \nu\,\frac{z'}{r}\,(rV)' - \left[\frac{(r/\alpha C)'}{(r/\alpha C)} + \nu\,\frac{r'}{r}\right](r\alpha p_H) - (r\alpha p_H)'\right\},$$

where $C = Eh$, $D = Eh^3/12(1-\nu^2)$, E is Young's modulus, ν is Poisson's ratio and h is the wall thickness of the shell. Note that the quantity $\mu\Phi$ is inversely proportional to the thickness h and so is large compared to unity if h is sufficiently small compared to other dimensions.

Quantities p_H and p_V denote radial and axial components of surface load intensity, while V is an axial stress resultant determined by overall static equilibrium. Normal stress resultants N_ξ and N_θ, transverse shear resultant Q, and stress couples M_ξ and M_θ are all defined in the usual way. The components of displacement of a point of the middle shell surface in the three coordinate directions r, θ, z are denoted by u, v, w, respectively, while β denotes the angle of rotation, due to deformation, of a tangent to a meridian. Although all other quantities are assumed to be independent of θ, the circumferential component of displacement v is allowed to have the form

$$v = kr\theta \qquad (k = \text{constant}) \tag{5}$$

which occurs when a sector of a shell is subjected to pure bending in a place perpendicular to the axis of revolution (see [8]). Once differential Eqs. (1) have been solved subject to appropriate boundary conditions, the various quantities may be determined from the following relations,

$$rV = -\int r\alpha p_V\,d\xi,$$

$$rN_\xi = r'\left(\frac{C}{r\alpha}\right)^{1/2}Y + \frac{z'}{\alpha}\,(rV), \qquad rQ = -z'\left(\frac{C}{r\alpha}\right)^{1/2}Y + \frac{r'}{\alpha}\,(rV),$$

$$\alpha N_\theta = \left(\frac{\alpha C}{r}\right)^{1/2}\left[Y' - \frac{1}{2}\,\frac{(r/\alpha C)'}{(r/\alpha C)}\,Y\right] + r\alpha p_H,$$

$$\alpha M_\xi = \left(\frac{\alpha D}{r}\right)^{1/2}\left\{X' + \left[\nu\,\frac{r'}{r} - \frac{1}{2}\,\frac{(rD/\alpha)'}{(rD/\alpha)}\right]X\right\}, \tag{6}$$

$$\alpha M_\theta = \left(\frac{\alpha D}{r}\right)^{1/2}\left\{\nu X' + \left[\frac{r'}{r} - \frac{\nu}{2}\,\frac{(rD/\alpha)'}{(rD/\alpha)}\right]X\right\},$$

$$\beta = \left(\frac{\alpha}{rD}\right)^{1/2}X, \qquad u = \frac{r}{C}\,(N_\theta - \nu N_\xi),$$

$$w = -\int\left[r'\beta - \frac{z'}{C}\,(N_\xi - \nu N_\theta)\right]d\xi.$$

3. Asymptotic integration when Φ has a first-order zero. Approximate solutions of differential equations (1) can often be obtained by methods of asymptotic integration provided the parameter μ in these equations is sufficiently large compared to unity. The form and completeness of the results depends on the nature of the coefficient functions Θ_1 , Θ_2 , and Φ. For instance, if these functions are all bounded and if Φ is also bounded from zero, asymptotic expansions in μ can be found which represent solutions of either the non-homogeneous differential equations or the corresponding homogeneous equations and which involve only elementary functions. Such expansions have been given for general shells of revolution by F. B. Hildebrand [5]. If Φ vanishes at some transition point or if Θ_1 and Θ_2 have at most second-order poles, as they usually do on the axis of revolution, asymptotic solutions can be obtained for the homogeneous differential equations using methods developed by R. E. Langer [6a, b] which involve Bessel functions. In general, Langer's methods yield only the leading terms of asymptotic expansions of solutions but may be used to find higher-order terms if the shell is such that Eqs. (1) can be reduced to a single second-order differential equation. Reduction of Eqs. (1) to a single second-order equation is possible whenever the shell thickness and the meridianal curvature are constant or, in general, whenever the ratio $(\Theta_1 - \Theta_2)/\Phi$ is constant, as has been shown recently by P. M. Naghdi [7] who also lists the form of Langer's asymptotic solutions.

The present paper is concerned with asymptotic solutions of non-homogeneous equations (1) assuming that all coefficient functions and loading terms are bounded functions of ξ for the shell or portion of shell being considered and that Φ vanishes at some isolated point. According to Eqs. (4), Φ vanishes whenever $z'(\xi)$ vanishes or whenever the tangent to a meridian is perpendicular to the axis of revolution. Define ξ so that $\xi = 0$ corresponds to the point where Φ is zero and let the range of ξ be restricted, if necessary, so that Φ is bounded from zero away from $\xi = 0$. Expanding about $\xi = 0$ we have

$$\mu\Phi(\xi) = [12(1 - \nu^2)]^{1/2}\left\{\xi\left[\frac{\alpha z''}{rh}\right]_{\xi=0} + \xi^2\left[\frac{\alpha z''}{2rh} + z''\left(\frac{\alpha}{rh}\right)'\right]_{\xi=0} + \cdots\right\}. \tag{7}$$

It follows that the coefficient of the first power of ξ in expansion (7) is different from zero only if $z''(0) \neq 0$ or only if the radius of curvature of a meridian is finite at $\xi = 0$. The function $\mu\Phi$ then is said to have a *first-order* zero and this is the only case we will consider. The parameter μ may be conveniently defined now as the value of the coefficient of ξ in (7). Using a zero subscript to denote the value of a quantity at $\xi = 0$ we may write

$$\mu = [12(1 - \nu^2)]^{1/2}\frac{\alpha_0 z_0''}{r_0 h_0} = [12(1 - \nu^2)]^{1/2}\frac{\alpha_0^3}{r_0 h_0 R_0}, \tag{8}$$

where R_0 is the radius of curvature of a meridian at $\xi = 0$. Dividing expansion (7) by formula (8) we now have

$$\Phi(\xi) = \xi + A_0\xi^2 + \cdots \tag{9}$$

where

$$A_0 = \frac{1}{2}\frac{z_0'''}{z_0''} + \frac{\alpha_0'}{\alpha_0} - \frac{r_0'}{r_0} - \frac{h_0'}{h_0}. \tag{10}$$

If there are no "rapid changes" in the shell surface or in its thickness, functions Θ_1 and Θ_2 are of order unity outside some neighborhood of the axis of revolution and so are small, generally, compared to $\mu\Phi$ provided μ is sufficiently large. This suggests that approximate solutions of differential equations (1) can be found by neglecting terms in Θ_1 and Θ_2 . The resulting differential equations may be interpreted as the real and imaginary parts of a single complex differential equation of the form

$$Z'' - i\mu\Phi Z = f \tag{11}$$

where

$$Z = X + iY, \qquad f = F + iG. \tag{12}$$

Thus, if Eqs. (1) cannot be reduced to a single second-order differential equation in an exact manner, they may be approximated by such an equation. The nature of the error in the approximation is discussed later on.

We consider now the problem of obtaining an approximate solution of differential equation (11) when μ is large. If we assume that a solution can be represented by an expansion in inverse powers of μ, we obtain the following formal solution

$$Z = \frac{if}{\mu\Phi} + \frac{1}{\mu^2\Phi}\left(\frac{f}{\Phi}\right)'' + \cdots \tag{13}$$

which obviously is not valid in some neighborhood of $\xi = 0$ where Φ vanishes. However, over an interval where Φ is bounded from zero a finite number of terms of (13) does furnish an approximate solution of (11). The first term of (13) will be called the *membrané* solution of (11), or of differential equations (1), since retaining only this term leads to the same results as the so-called *membrane* theory of shells. An approximate solution of Eq. (11) which is valid at $\xi = 0$ and which reduces to the membrane solution away from $\xi = 0$ may be constructed as follows. First, in the immediate neighborhood of $\xi = 0$ differential equation (11) may be approximated by

$$Z'' - i\mu\xi Z = f(0), \tag{14}$$

where we assume that $f(0) \neq 0$. A solution of (14) is given by $Z = \mu^{-2/3} f(0) T(\mu^{1/3}\xi)$ provided the function $T(x)$ is a solution of

$$T''(x) - ixT(x) = 1. \tag{15}$$

(Henceforth, primes are used to denote differentiation with respect to the written or implied argument of a function.) Let $T(x)$ stand for that (unique) solution of (15) which has the behavior $T(x) \sim i/x$ as $| x | \to \infty$ and consider the expression

$$Z_1(\xi) = \mu^{-2/3} f(\xi) T(\mu^{1/3}\Phi). \tag{16}$$

Near $\xi = 0$, $\Phi \sim \xi$ and $f(\xi) \sim f(0)$ so that Z_1 approximates a solution of (14) or (11). Away from $\xi = 0$, $\mu^{1/3}\Phi$ is large for large μ so that $Z_1 \sim if/\mu\Phi$. Consequently, Z_1 furnishes an "interpolation" between a solution of Eq. (14) and the membrane solution of (11). The real and imaginary parts of the required function $T(x)$ and its derivative are tabulated briefly in [2]. Also the following representations hold,

$$T(x) = -\int_0^\infty \exp(-ixt - \tfrac{1}{3}t^3)\,dt$$

$$= -\sum_{n=0}^{\infty} (-i)^n 3^{(n-2)/3}\Gamma\left(\frac{n+1}{3}\right)\frac{x^n}{n!} \tag{17}$$

$$\sim \sum_{n=0}^{\infty} \frac{i^{3n+1}(3n)!}{3^n n! x^{3n+1}}, \qquad |x| >> 1.$$

It can be shown that the real and imaginary parts of Z_1 are truly *asymptotic* solutions of differential equations (1) in the sense that the relative error approaches zero uniformly as $\mu \to \infty$. Even so, Z_1 may not yield sufficiently accurate results when μ is only moderately large making it desirable to have higher-order approximations to solutions of Eqs. (1) or (11). Recently [3] complete asymptotic expansions were found which represent a solution of a non-homogeneous second-order differential equation such as (11). They are of the form

$$Z_2(\xi) = \mu^{-2/3} \sum_{n=0}^{\infty} \mu^{-n/3} \sum_{k=0}^{[2n/3]} a_{2n,2n-3k}(\xi) T^{(2n-3k)}(x), \tag{18}$$

$$Z_3(\xi) = \mu^{-2/3} T(x) \sum_{n=0}^{\infty} \mu^{-n} a_n(\xi) + \mu^{-1} \sum_{n=0}^{\infty} \mu^{-n} b_n(\xi) + \mu^{-4/3} T'(x) \sum_{n=0}^{\infty} \mu^{-n} c_n(\xi), \tag{19}$$

where $T^{(k)}(x)$ denotes the kth derivative of T with respect to x and

$$x = \mu^{1/3}(3\omega/2)^{2/3}, \qquad \omega = \int_0^\xi \Phi^{1/2}(\xi)\,d\xi. \tag{20}$$

The first term of expansion (18) is similar to the expression for Z_1, while expansion (19) is analogous to expansions obtained by Langer [6c] for solutions of a homogeneous second-order differential equation. The various coefficients in either expansion may be successively determined so that the expansion formally satisfies differential equation (11). The expansions do not converge, in general, but a finite number of terms of either furnishes an approximate solution with an error of the order in μ of the first term neglected. Depending on the circumstances one expansion may be easier to use than the other but one may also interpret expansion (19) as simply a "rearrangement" of the terms in (18) accomplished by using differential equation (15) to express higher derivatives of T in terms of T and T'. If Eq. (11) is only an approximation of Eq. (1), it would not be consistent to retain arbitrarily many terms in Z_2 or Z_3, considered now as approximate solutions of differential equation (11). In [3] it is shown that terms of relative order $\mu^{-2/3}$ in either Z_2 or Z_3 depend on terms such as those in Θ_1 and Θ_2 in differential equations (1). This fact leads to the result that when μ is large the relative error in approximating a solution of Eqs. (1) by a solution of Eq. (11) is of the order $\mu^{-2/3}$, making it generally inconsistent to retain more than the first two terms in the expansions for Z_2 and Z_3. But if Eqs. (1) can be reduced to a second-order equation in an exact manner, then theoretically asymptotic solutions with arbitrarily many terms can be found.

For later reference we now list the explicit form of the first three terms in the two expansions for a solution of differential equation (11), namely

$$Z_2(\xi) = \frac{1}{\mu^{2/3}} fQ^4T(x) + \frac{i}{\mu} Q^3(Q^3f)'T''(x) + \frac{i}{\mu^{4/3}} \{Q^7Q''fT'(x) - \tfrac{1}{2}Q^3[Q^2(Q^3f)']'xT''(x)\}, \tag{21}$$

$$Z_3(\xi) = \frac{1}{\mu^{2/3}} f(0)QT(x) + \frac{i}{\mu}\left[\frac{f}{\Phi} - \frac{f(0)}{Q^3\Phi}\right] + \frac{i}{\mu^{4/3}} \frac{f(0)T'(x)}{2Q\Phi^{1/2}} \int_0^\xi \frac{Q''}{Q\Phi^{1/2}}\, d\xi, \tag{22}$$

where

$$Q(\xi) = (3\omega/2)^{1/6}/\Phi^{1/4}, \qquad Q(0) = 1. \tag{23}$$

Note that definitions (23) and (20) imply $x = \mu^{1/3}\Phi Q^4$ and also that $dx/d\xi = \mu^{1/3}/Q^2$. If $T(x)$ is replaced by its asymptotic form i/x when x is large both (21) and (22) reduce to the membrane solution $if/\mu\Phi$ plus terms of the same order as the second term in expansion (13). When $x = O(1)$ it follows that $\xi = O(\mu^{-1/3})$. Then if f, Φ, and Q are expanded in powers of ξ and terms of relative order $\mu^{-2/3}$ are neglected, expressions (21) and (22) both reduce to

$$Z = \frac{f(0)}{\mu^{2/3}} T(\mu^{1/3}\xi) + \frac{1}{\mu}\left\{\frac{A_0}{5} f(0)[\mu^{2/3}\xi^2T'(\mu^{1/3}\xi) - \mu^{1/3}\xi T(\mu^{1/3}\xi)] + i[f'(0) - \tfrac{3}{5}A_0f(0)]\right\}, \tag{24}$$

where A_0 is defined by Eq. (10). Although expression (16) for Z_1 also reduces to the membrane solution when x is large, for $x = O(1)$ it reduces to an expression that agrees with only the first term of (24), showing that the relative error in Z_1 is of the order $\mu^{-1/3}$, in general.

The asymptotic solutions described above are applied to shell problems in the following sections.

4. Toroidal shells of arbitrary cross section. We shall consider two particular problems involving toroidal shells. One problem is the bending of a curved tube assuming that the tube forms a sector of a shell of revolution. The other is the deflection of a toroidal expansion joint due to an axial force, where the expansion joint consists of a toroidal shell slit around its inner circumference and joined at the edges to sections of

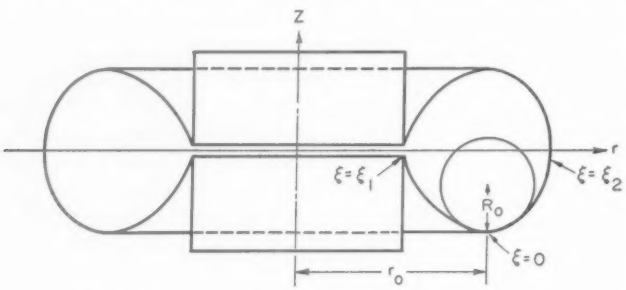

FIG. 1. Cross section of toroidal-shell expansion joint.

cylindrical pipe (see Fig. 1). As will be seen, the two problems are mathematically quite similar. Asymptotic solutions of both have been given previously for toroidal shells of circular cross section and uniform thickness [2, 4]. For the present we assume only that the shell has a plane of symmetry perpendicular to the axis of revolution.

If P denotes the axial force applied to the expansion joint the loading terms in differential equations (1) become

$$F = r'\left(\frac{\alpha}{rD}\right)^{1/2}\frac{P}{2\pi}, \qquad G = \left[\frac{r'z'}{r^2} + \nu\frac{(z'/\alpha C)'}{(r/\alpha C)}\right]\left(\frac{r}{\alpha C}\right)^{1/2}\frac{P}{2\pi}. \tag{25}$$

Generally G is small compared to F and if coefficients Θ_1 and Θ_2 are neglected in Eqs. (1) it is consistent to neglect G also. The right side of differential equation (11) is then given by the first of Eqs. (25).

For the problem of a tube or sector of a shell of revolution bent by end moments applied in a plane perpendicular to the axis of revolution we have

$$F = 0, \qquad G = r'\left(\frac{\alpha C}{r}\right)^{1/2}k. \tag{26}$$

The constant k is the same as that in Eq. (5) and is related to the change in curvature ΔK of the neutral surface of a tube by $k = r_n\,\Delta K$, where r_n is the radius of the cylindrical neutral surface. For tubes with closed cross section the applied moment m is given by

$$m = \int (r - r_n)N_\theta\alpha\,d\xi. \tag{27}$$

After differential equations (1) have been solved and stress resultants obtained in terms of k, the relationship between k or ΔK and the applied moment m may be established using (27).

Let the plane $z = 0$ be the plane of symmetry for either problem and consider only the lower half of the shell, as shown in Fig. 1. Let ξ_1 and ξ_2 correspond to the inner and outer points, respectively, where a meridian meets the plane of symmetry. For the tube bending problem it follows from symmetry that $\beta = 0$ and $Q = 0$ at both ξ_1 and ξ_2. These conditions also hold at ξ_2 for the expansion joint but may not be the actual edge conditions at ξ_1. However, we shall assume that $\beta = 0$ and $Q = 0$ at ξ_1 for the expansion joint also, since it can be shown that axial deflection and maximum stresses are not significantly affected by a change in the conditions at ξ_1. Then, according to Eqs. (6) and (12), the boundary conditions for either problem are equivalent to

$$Z(\xi_1) = 0, \qquad Z(\xi_2) = 0. \tag{28}$$

The closed meridians of the toroidal shell are assumed to have continuously turning tangents which implies that r' vanishes at ξ_1 and ξ_2. Since the assumed loading term $f(\xi)$ for either problem involves r' as a factor, it follows that the membrane solution $if/\mu\Phi$ vanishes at ξ_1 and ξ_2 so that asymptotic solutions Z_1, Z_2, and Z_3 all satisfy boundary conditions (28), at least to within terms of the order of the error involved. This means that no solution of the corresponding homogeneous differential equation is required and that approximate expressions for the various stress resultants and couples and displacements can be obtained from the real and imaginary parts of Z_1, Z_2, or Z_3 together with Eqs. (6). To evaluate the stresses and displacements as functions of ξ one must first tabulate the functions Φ, Q, and x given by Eqs. (4), (20), and (23). These functions

in turn depend on the exact shape of a meridian of the middle surface and on any variation in the shell thickness. Nevertheless, maximum stresses and other quantities of interest can be obtained explicitly without assuming any specific shape for the shell cross section.

Since $T(x) \sim i/x$ for large x the most significant values of Z as given by (16), (21), or (22) occur where $x = O(1)$ or $\xi = O(\mu^{-1/3})$. For this range of ξ expression (24) may be used as an approximate solution of differential equation (11) with a relative error of order $\mu^{-2/3}$. When the real and imaginary parts of (24) are substituted in Eqs. (6) it becomes apparent that for large values of μ the most significant stresses are the circumferential direct stress $\sigma_{\theta D}$ and the meridianal bending stress $\sigma_{\xi B}$, defined by

$$\sigma_{\theta D} = \frac{N_\theta}{h}, \qquad \sigma_{\xi B} = \frac{6M_\xi}{h^2}. \tag{29}$$

The maximum numerical values of these stresses and their location can be determined in a straightforward way using the following information about the real and imaginary parts of $T(x)$, denoted by $T_r(x)$ and $T_i(x)$. The maximum value of $T_i'(x)$ occurs at $x = 0$ where one has

$$T_r(0) = -1.288, \qquad T_i(0) = 0, \qquad T_r'(0) = 0, \qquad T_i'(0) = .939. \tag{30}$$

The maximum value of $T_r'(x)$ occurs at $x = 1.225$ where

$$T_r(1.225) = -.692, \quad T_i(1.225) = .817, \quad T_r'(1.225) = .753, \quad T_i'(1.225) = .214. \tag{31}$$

Also, $T_r(x)$ is an even function of x, $T_i(x)$ an odd function of x.

For the expansion joint problem we find after various calculations that

$$\begin{aligned} \sigma_{\theta D,\max} &= \sigma_{\theta D}(\xi_D) = .342(1-\nu^2)^{1/3}\left(\frac{R_0 h_0}{r_0^2}\right)^{1/3} \frac{P}{h_0^2}\left[1 + O\left(\frac{1}{\mu^{2/3}}\right)\right] \\ \sigma_{\xi B,\max} &= \sigma_{\xi B}(\xi_B) = \pm \frac{.475}{(1-\nu^2)^{1/6}}\left(\frac{R_0 h_0}{r_0^2}\right)^{1/3} \frac{P}{h_0^2}\left[1 \pm \frac{2.14A_1 + .92A_2}{\mu^{1/3}} + O\left(\frac{1}{\mu^{2/3}}\right)\right], \end{aligned} \tag{32}$$

where

$$\begin{aligned} \xi_D &= -\frac{.73}{\mu^{2/3}}\frac{r_0'}{r_0} + O\left(\frac{1}{\mu}\right), \\ \xi_B &= \pm\frac{1.225}{\mu^{1/3}} + \frac{1}{\mu^{2/3}}(.30A_0 - .70A_2) + O\left(\frac{1}{\mu}\right) \end{aligned} \tag{33}$$

with

$$\begin{aligned} A_1 &= \frac{1}{10}\frac{z_0'''}{z_0''} - \frac{7}{10}\frac{r_0'}{r_0} - \frac{3}{10}\frac{\alpha_0'}{\alpha_0} - \frac{7}{10}\frac{h_0'}{h_0}, \\ A_2 &= (1-\nu)\frac{r_0'}{r_0} + 2\frac{h_0'}{h_0}. \end{aligned} \tag{34}$$

As before a zero subscript denotes the value of a quantity at $\xi = 0$. The plus and minus signs in the formula for $\sigma_{\xi B}$ correspond, respectively, to the signs in ξ_B and the constant A_0 in ξ_B is defined by Eq. (10).

The total axial deflection or spread of the expansion joint due to bending only is given by

$$\delta = 2[w(\xi_2) - w(\xi_1)] = -2\int_{\xi_1}^{\xi_2} r'\beta \, d\xi. \tag{35}$$

Since the integral extends over the full range of ξ, expression (24) is no longer appropriate. If the first two terms of Z_3 are used instead and β is determined according to Eqs. (6), the expression for δ takes the form

$$\delta = -\frac{r_0'P}{\pi\mu}\left(\frac{\alpha_0}{r_0D_0}\right)^{1/2}\int_{x_1}^{x_2} r'\left(\frac{\alpha}{rD}\right)^{1/2} Q^3T_r(x)\, dx, \tag{36}$$

where x_1 and x_2 are values of x corresponding to ξ_1 and ξ_2. Since $T_r(x) \sim 2/x^4$ for large x, the major contribution of the integrand in (36) occurs for $x = 0(1)$ or $\xi = 0(\mu^{-1/3})$. Also, interpreting $T(x)$ as a Fourier transform according to the integral representation given in (17) it follows that $\int_0^\infty T_r(x) = -\pi/2$. Using these facts and introducing the explicit form of μ as defined in (8) it can be shown that

$$\delta = [12(1-\nu^2)]^{1/2}\frac{R_0P}{Eh_0^2}\left[1 + O\left(\frac{1}{\mu^{2/3}}\right)\right]. \tag{37}$$

This expression and the leading terms in expressions (32) are exactly the same as the corresponding results for a toroidal shell with a circular cross section of radius R_0 and a mean shell radius r_0, showing that when μ is sufficiently large the axial deflection and maximum stresses depend on the exact shape of the cross section only near the point where the slope dz/dr of a meridian is zero.

For the tube bending problem results similar to (32) can be obtained with Ek in place of P/h_0^2. However, it is more convenient to obtain first the relationship between the applied moment m and the constant k. Substituting for αN_θ from Eqs. (6), with Y taken as the imaginary part of Z_3 when $f = ikr'(\alpha C/r)^{1/2}$, and integrating by parts, finally reduces formula (27) to

$$m = -\frac{2kr_0'}{\mu}\left(\frac{\alpha_0C_0}{r_0}\right)^{1/2}\int_{x_1}^{x_2} r'\left(\frac{\alpha C}{r}\right)^{1/2} Q^3T_r(x)\, dx, \tag{38}$$

which is analogous to expression (36). Evaluating the integral in (38) in the same way as in (36) and introducing μ from Eq. (8), yields the result,

$$m = \frac{2\pi}{[12(1-\nu^2)]^{1/2}} R_0h_0^2Ek\left[1 + O\left(\frac{1}{\mu^{2/3}}\right)\right]. \tag{39}$$

We recall that $k = r_n\,\Delta K$, where ΔK is the change in curvature of the neutral surface and r_n is the radius of the neutral surface. Writing $r_n = r(\xi_n)$ the value of ξ_n is by definition a solution of the equation $N_\theta - \nu N_\xi = 0$. From expression (24) and Eqs. (6) one obtains

$$\xi_n = -\frac{1.288}{\mu^{2/3}}\left[A_1 + (1+\nu)\frac{r_0'}{r_0}\right] + O\left(\frac{1}{\mu}\right), \tag{40}$$

where A_1 is given by Eqs. (34). It follows from (40) that $r_n = r_0[1 + 0(\mu^{-2/3})]$, so that expression (39) can also be written in the form

$$m = \frac{2\pi}{[12(1-\nu^2)]^{1/2}} r_0R_0h_0^2E\Delta K\left[1 + O\left(\frac{1}{\mu^{2/3}}\right)\right]. \tag{41}$$

This result may be compared with the equation $m = EI\,\Delta K$ which holds for a straight tube according to elementary beam theory, where I is the moment of inertia of the cross section about an axis through the centroid. Writing

$$m = \rho EI\Delta K \tag{42}$$

for a curved tube serves to define the quantity ρ as a *rigidity factor* which has the value unity for a straight tube. Comparing (41) and (42) it follows that

$$\rho = \frac{2\pi}{[12(1-\nu^2)]^{1/2}}\frac{r_0R_0h_0^2}{I}\left[1+O\left(\frac{1}{\mu^{2/3}}\right)\right] = \frac{2}{\mu}\frac{\pi\alpha_0^3h_0}{I}\left[1+O\left(\frac{1}{\mu^{2/3}}\right)\right]. \tag{43}$$

Formula (43) is remarkably simple and shows that when μ is sufficiently large the *effective stiffness* ρEI of a curved tube depends on only the dimensions r_0, R_0 and h_0 associated with the point of the cross section where $\xi = 0$. For a tube with circular cross section of radius R_0 and uniform thickness h_0, $I = \pi R_0^3h_0$. Also the quantity α may be identified with R_0 so that the rigidity factor then becomes simply $\rho = 2/\mu$ when μ is large.

The maximum stresses which occur in the bending of a curved tube may be compared with the maximum stress in a straight tube of circular cross section with radius R_0 and uniform thickness h_0 given by

$$\sigma_{SM} = \frac{mR_0}{\mathrm{I}} = \frac{m}{\pi R_0^2h_0}. \tag{44}$$

Introducing the expression for m given in Eq. (39) and determining the maximum stresses in the same way as in the expansion joint problem we find

$$\frac{\sigma_{\theta D,\max}}{\sigma_{SM}} = \frac{\sigma_{\theta D}(\xi_D)}{\sigma_{SM}} = \pm\,.862(1-\nu^2)^{1/3}\left(\frac{R_0^2}{r_0h_0}\right)^{2/3}\left[1 \pm \frac{2.14A_1 + .92(r_0'/r_0)}{\mu^{1/3}} + O\left(\frac{1}{\mu^{2/3}}\right)\right] \tag{45}$$

$$\frac{\sigma_{\xi B,\max}}{\sigma_{SM}} = \frac{\sigma_{\xi B}(\xi_B)}{\sigma_{SM}} = -\frac{1.86}{(1-\nu^2)^{1/6}}\left(\frac{R_0^2}{r_0h_0}\right)^{2/3}\left[1+O\left(\frac{1}{\mu^{2/3}}\right)\right],$$

where now

$$\xi_D = \pm\frac{1.225}{\mu^{1/3}} + \frac{1}{\mu^{2/3}}\left(.30A_0 - .70\,\frac{r_0'}{r_0}\right) + O\left(\frac{1}{\mu}\right)$$
$$\xi_B = -\frac{.729}{\mu^{2/3}} + O\left(\frac{1}{\mu}\right). \tag{46}$$

The constant A_0 is defined by Eq. (10), A_1 and A_2 by Eqs. (34). The leading terms in (45) are the same as those for a tube with circular cross section of radius R_0. Thus, for sufficiently large μ the maximum stresses due to bending of a curved tube of arbitrary cross section can be obtained by considering an "equivalent" tube of circular cross section.

5. Toroidal shells of circular cross section. We now consider the form which the results in the previous section take for toroidal shells or curved tubes of circular cross section and uniform thickness h. We shall also list additional terms of relative order $\mu^{-2/3}$.

For shells of the special type now being considered parametric equations (2) may be taken as

$$r = a + b \sin \xi, \qquad z = -b \cos \xi, \tag{47}$$

so that $r_0 = a$, $R_0 = b$, $\alpha = b$. The parameter μ and a new parameter λ are given by

$$\lambda = \frac{b}{a}, \qquad \mu = [12(1 - \nu^2)]^{1/2} \frac{b^2}{ah}. \tag{48}$$

Previous asymptotic solutions of the special problems being discussed did not take into account the effect of terms in λ which was assumed to be negligibly small.

For the expansion-joint problem the largest values of the meridianal bending stress $\sigma_{\xi B}$ are now given by

$$\sigma_{\xi B}(\xi_B) = \pm .482 \left(\frac{bh}{a^2}\right)^{1/3} \frac{P}{h^2} \left[1 \mp .86 \frac{\lambda}{\mu^{1/3}} + O\left(\frac{1}{\mu^{2/3}}\right)\right], \tag{49}$$

where we have taken $\nu = .3$ and where

$$\xi_B = \pm \frac{1.225}{\mu^{1/3}} - .79 \frac{\lambda}{\mu^{2/3}} + O\left(\frac{1}{\mu}\right). \tag{50}$$

It follows that the largest of the two values listed occurs at the "inner" point where ξ_B is negative. For the tube-bending problem the only stress formula which involves λ is the following for the maximum circumferential direct stress,

$$\sigma_{\theta D}(\xi_D) = \pm .377 \mu^{2/3} \sigma_{SM} \left[1 \mp .58 \frac{\lambda}{\mu^{1/3}} + O\left(\frac{1}{\mu^{2/3}}\right)\right], \tag{51}$$

where again we have taken $\nu = .3$ and where

$$\xi_D = \pm \frac{1.225}{\mu^{1/3}} - .997 \frac{\lambda}{\mu^{2/3}} + O\left(\frac{1}{\mu}\right). \tag{52}$$

From Eqs. (49) and (52) it is easily seen that terms in λ play a minor role when λ is small compared to $\mu^{1/3}$, which does not necessarily mean that λ is small compared to unity as had been assumed in earlier work. However, the correction due to retaining terms in λ may certainly be significant. For instance, when $\lambda = 1/4$ and $\mu = 8$ the correction to the maximum bending stress in the expansion joint, as given by Eq. (49), amounts to 10 per cent.

The relative size of the correction terms in λ given above does not furnish any estimate of the accuracy of asymptotic formulas since at least part of the error term is independent of λ. Theoretically, exact expressions can be obtained for the terms of relative order $\mu^{-2/3}$ (or higher) but the necessary calculations become quite involved. However, if all terms in λ are neglected higher-order terms in μ can be obtained without great effort. Neglecting all terms in λ is equivalent to approximating differential equations (1) by the single complex equation

$$Z'' - i\mu \sin \xi Z = f_0 \cos \xi, \tag{53}$$

where $f_0 = (b^3/aD)^{1/2} P/2\pi$ for the expansion-joint problem and $f_0 = i(b^3hE/a)^{1/2} k$ for the tube-bending problem. Neglecting terms in λ is also equivalent to assuming that the various stresses and displacements are symmetrical with respect to $\xi = 0$. An approximate asymptotic solution of (53) which includes terms of relative order $\mu^{-2/3}$ is given by either expression (21) or (22). To determine the most significant stresses one again

restricts attention to the range where $\xi = 0(\mu^{-1/3})$ and expands all functions in powers of ξ. In this way the following results are finally obtained.

For the expansion-joint problem

$$\sigma_{\theta D,\max} = \sigma_{\theta D}(0) = .342(1-\nu^2)^{1/3}\left(\frac{bh}{a^2}\right)^{1/3}\frac{P}{h^2}\left[1 - \frac{.342}{\mu^{2/3}} + O\left(\frac{1}{\mu}\right)\right],$$
$$\sigma_{\xi B,\max} = \sigma_{\xi B}(\xi_M) = \pm\frac{.475}{(1-\nu^2)^{1/6}}\left(\frac{bh}{a^2}\right)^{1/3}\frac{P}{h^2}\left[1 - \frac{.054}{\mu^{2/3}} + O\left(\frac{1}{\mu}\right)\right], \tag{54}$$

$$\xi_M = \pm\left[\frac{1.225}{\mu^{1/3}} - \frac{.017}{\mu} + O\left(\frac{1}{\mu^{4/3}}\right)\right], \tag{55}$$

$$\delta = [12(1-\nu^2)]^{1/2}\frac{bP}{Eh^2}\left[1 + O\left(\frac{1}{\mu}\right)\right]. \tag{56}$$

Analogously, for the tube-bending problem

$$\sigma_{\theta D,\max} = \sigma_{\theta D}(\xi_M) = \pm.377\mu^{2/3}\sigma_{SM}\left[1 - \frac{.054}{\mu^{2/3}} + O\left(\frac{1}{\mu}\right)\right],$$
$$\sigma_{\xi B,\max} = \sigma_{\xi B}(0) = -\frac{.814}{(1-\nu^2)^{1/2}}\mu^{2/3}\sigma_{SM}\left[1 - \frac{.342}{\mu^{2/3}} + O\left(\frac{1}{\mu}\right)\right], \tag{57}$$

where ξ_M is again given by Eq. (55). Note that there is no "correction" term of order $\mu^{-2/3}$ in formula (56) for the axial deflection δ indicating that in an exact solution this term is probably proportional to λ. For the bending of a curved tube the analogous result for the rigidity factor is $\rho = (2/\mu)\,[1 + 0(1/\mu)]$.

In [4] where the leading terms in the above results for the tube-bending problem were first obtained, the asymptotic formulas are compared with numerical results calculated by L. Beskin [1] using a strain-energy principle in a manner equivalent to finding an exact trigonometric series solution of differential equation (53). In Tables 1 and 2 we have listed the numerical values obtained by retaining either one or two terms in formulas (57) and also the values obtained by Beskin.

TABLE 1.

Values of $\sigma_{\theta D,\max}/\sigma_{SM}$.

μ	$.377\mu^{2/3}$	$.377\mu^{2/3}-.020$	Beskin [1]
1.00	.377	.357	.979
2.50	.694	.674	.889
3.54	.875	.855	.881
5.00	1.10	1.08	1.010
7.07	1.39	1.37	1.277
10.00	1.75	1.73	1.680
12.50	2.03	2.01	1.99
17.68	2.56	2.54	2.54
25.00	3.22	3.20	3.26
35.36	4.06	4.04	4.08
50.00	5.11	5.09	5.09
70.71	6.45	6.43	6.43
100.00	8.13	8.11	8.12

TABLE 2.

Values of $-\sqrt{1-\nu^2}\,\sigma_{\xi B,\max}/\sigma_{SM}$.

μ	$.814\mu^{2/3}$	$.814\mu^{2/3}-.278$	Beskin [1]
1.00	.814	.536	.432
2.50	1.50	1.22	1.06
3.54	1.89	1.61	1.47
5.00	2.38	2.10	2.01
7.07	3.00	2.72	2.69
10.00	3.78	3.50	3.51
12.50	4.38	4.10	4.11
17.68	5.52	5.24	5.25
25.00	6.96	6.68	6.67
35.36	8.77	8.49	8.42
50.00	11.0	10.7	10.7
70.71	13.9	13.6	13.5
100.00	17.5	17.2	17.2

We observe that for $\sigma_{\theta D,\max}$ the values found using the two-term asymptotic formula coincide with the values obtained by Beskin for $\mu \geq 50$ and differ from Beskin's results by less than 3 per cent for $\mu \geq 10$. Formula (55) for the location of the maximum yields values which differ from those given by Beskin by less than 2 per cent for $\mu \geq 5$, and by less than 1 per cent for $\mu > 10$, except for $\mu = 50$ when there is a difference of more than 3 per cent.

From Table 2 for $\sigma_{\xi B,\max}$ we see that although the one-term asymptotic formula does not agree as closely with Beskin's results as does the one-term formula for $\sigma_{\theta D,\max}$, the two-term formula yields values which differ from those found by Beskin by 1 per cent or less for $\mu \geq 7$, which is an even closer agreement than was found for $\sigma_{\theta D,\max}$. In fact, the two-term asymptotic formula for $\sigma_{\xi B,\max}$ differs from Beskin's results by less than 5 per cent for μ as small as 5.

Finally, it is of some interest to compare expressions (49) and (54). Unless $\lambda < .1/\mu^{1/3}$ the correction to the one-term asymptotic solution due to retaining terms in λ may be much more significant than that due to terms of relative order $\mu^{-2/3}$, at least as far as the dominant stress $\sigma_{\xi B}$ for the expansion-joint problem is concerned. A similar remark can be made about the circumferential direct stress $\sigma_{\theta D}$ for the tube-bending problem, although this stress is not the dominant one for this problem when μ is large. Also, the "λ-correction" to ξ_B or ξ_D in formulas (50) and (52) may be considerably more important than the second term in formula (55).

References

1. L. Beskin, *Bending of curved thin tubes*, J. Appl. Mechanics, **12**, A1-A7 1945)
2. R. A. Clark, *On the theory of thin elastic toroidal shells*, J. Math. and Phys. **29**, 146-178 (1950)
3. R. A. Clark and W. E. Gibson, *Asymptotic solution of a non-homogeneous differential equation*, in preparation.
4. R. A. Clark and E. Reissner, *Bending of curved tubes*, Advances Appl. Mech. **II**, Academic Press, New York, 1951, 93-122
5. F. B. Hildebrand, *On asymptotic integration in shell theory*, Proc. Symposia Appl. Math. **3**, McGraw-Hill, New York 1950, 53-66

6. R. E. Langer, *On the asymptotic solution of ordinary differential equations*, Trans. Am. Math. Soc.(a) **33**, 22-64 (1931); (b) **37**, 397-416 (1935), (c) **67**, 461-490 (1949)
7. P. M. Naghdi, *On the deformation of elastic shells of revolution*, Quart. Appl. Math. **12**, 369-374 (1954)
8. E. Reissner, *On bending of curved thin-walled tubes*, Proc. Natl. Acad. Sci. U. S. **35**, 204-208 (1949)
9. E. Reissner, *On the theory of thin elastic shells*, Reissner Anniversary Volume, Edwards, Ann Arbor, 1949, 231-247

ON LIMIT ANALYSIS OF PLATES*

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Summary. The bending of thin perfectly plastic plates of arbitrary shape under transverse load is studied. The collapse load for a concentrated force on such a plate is found to be 2π times the yield moment.

1. Introduction. In the following an attempt is made to generalize the work of Hopkins and Prager [1]† on the load-carrying capacities of circular plates to non-symmetrical cases. The basic equations for this purpose have been given previously by Hopkins [2] for a plate composed of a material obeying Tresca's yield criterion. We now discuss in detail the different types of plastic regimes that can arise. After a brief consideration of the discontinuities in the moment field, the technique of limit analysis [3] is used to determine the limit load for a concentrated force. Upper bounds on the limit loads in non-symmetrical cases have also been obtained by Rzhanitsyn [4]. Only in very artificial examples has it been found possible to obtain the actual moment distribution and deformation mode during collapse.

2. Equations of equilibrium in curvilinear coordinates. Let M_1 , M_2 be the principal moments; Q_1 , Q_2 the shearing forces; p the load per unit area; ρ_1 , ρ_2 the radii of curvature and s_1 , s_2 the arc-lengths of the principal stress trajectories (Fig. 1). Then the equations of equilibrium are

$$\begin{aligned} \frac{\partial M_1}{\partial s_1} + \frac{M_1 - M_2}{\rho_2} - Q_1 &= 0, \\ \frac{\partial M_2}{\partial s_2} + \frac{M_1 - M_2}{\rho_1} - Q_2 &= 0, \\ \frac{\partial Q_1}{\partial s_1} + \frac{\partial Q_2}{\partial s_2} + \frac{Q_1}{\rho_2} - \frac{Q_2}{\rho_1} &= -p. \end{aligned} \tag{1}$$

We now introduce orthogonal curvilinear coordinates u, v such that the principal stress trajectories are given by $u = \text{const.}$ and $v = \text{const.}$ Let

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \tag{2}$$

be the first fundamental form of the net of stress trajectories.** We have then

$$\begin{aligned} F = 0, \qquad \frac{\partial}{\partial s_1} &= \frac{1}{E^{1/2}}\frac{\partial}{\partial u}, \qquad \frac{\partial}{\partial s_2} = \frac{1}{G^{1/2}}\frac{\partial}{\partial v}, \\ \frac{1}{\rho_1} = \frac{\partial \varphi}{\partial s_1} &= \frac{\varphi_u}{E^{1/2}} = \frac{-E_v}{2E\,G^{1/2}}, \\ \frac{1}{\rho_2} = \frac{\partial \varphi}{\partial s_2} &= \frac{\varphi_v}{G^{1/2}} = \frac{G_u}{2G\,E^{1/2}}, \end{aligned} \tag{3}$$

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†Numbers in square brackets refer to the Bibliography at the end of the paper.

**Because $F = 0$, it is not necessary to use here the tensor notation g_{ij} .

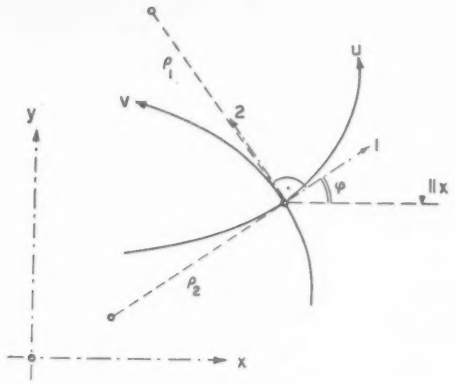

FIG. 1.

where φ is the angle of inclination of the curves $v = \text{const.}$ to the x-axis, and subscripts denote partial differentiation. Further, it is convenient to introduce the quantities $\omega = (M_1 + M_2)/2$ and $\delta = (M_1 - M_2)/2$. The first two of Eqs. (1) are then

$$\begin{aligned} \frac{(\omega + \delta)_u}{E^{1/2}} + \frac{2\delta \cdot \varphi_v}{G^{1/2}} - Q_1 &= 0, \\ \frac{(\omega - \delta)_v}{G^{1/2}} + \frac{2\delta \cdot \varphi_u}{E^{1/2}} - Q_2 &= 0, \end{aligned} \tag{4}$$

and the third equation of equilibrium becomes

$$\frac{(Q_1)_u}{E^{1/2}} + \frac{(Q_2)_v}{G^{1/2}} + Q_1 \frac{G_u}{2G\,E^{1/2}} + Q_2 \frac{E_v}{2E\,G^{1/2}} = -p. \tag{5}$$

To eliminate Q_1 , Q_2 we introduce (4) in (5) and get

$$\begin{aligned} \frac{\omega_{uu}}{E} + \frac{\omega_{vv}}{G} + \frac{1}{(EG)^{1/2}} \left[\left(\frac{G}{(EG)^{1/2}} \right)_u \omega_u + \left(\frac{E}{(EG)^{1/2}} \right)_v \omega_v \right] + \frac{\delta_{uu}}{E} - \frac{\delta_{vv}}{G} \\ + \frac{1}{(EG)^{1/2}} \left[\left(\frac{G}{(EG)^{1/2}} \right)_u \delta_u - \left(\frac{E}{(EG)^{1/2}} \right)_v \delta_v \right] \\ + 4 \frac{\delta}{(EG)^{1/2}} \varphi_{uv} + 2 \left[\frac{\varphi_v}{(EG)^{1/2}} \delta_u + \frac{\varphi_u}{(EG)^{1/2}} \delta_v \right] = -p. \end{aligned} \tag{6}$$

3. The yield condition. Tresca's yield condition is shown in Fig. 2. For this condition only four essentially different types of regimes occur, namely two stress determined regimes, represented for example by the points A and B, and two kinematically determined regimes, represented for example by the sides AB and BC of the yield hexagon.

Regime A. As $\delta = 0$, the angle φ is indeterminate. This regime can occur in a finite region only if the pressure p is zero (see also Prager [5]).

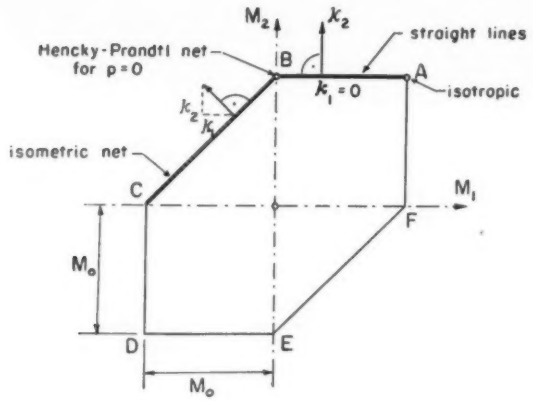

FIG. 2.

Regime B. In this regime $\omega = M_0/2$ and $\delta = -M_0/2$, where M_0 is the yield moment. Equation (6) becomes

$$2M_0\varphi_{us} = p\,(EG)^{1/2}. \tag{7}$$

For the special case $p = 0$, we have

$$\varphi_{us} = 0. \tag{8}$$

This is the condition that the stress trajectories form a Hencky-Prandtl-net [6].

Regime AB. For this regime it is necessary to consider the velocity field. The principal curvature-rates are denoted by κ_1 and κ_2 . For moment states represented by points on the side AB, the flow rule requires $\kappa_1 = 0$. Thus the surface into which the plate is deforming has *parabolic* curvature, and therefore one set of the principal lines of curvature is *straight* (see for example, Blaschke [7], p. 112). Because of the isotropy condition it follows that one family of the principal stress trajectories consists of straight lines. As $\omega - \delta = M_2 = M_0$ the second of Eqs. (1) becomes

$$Q_2 = 0. \tag{9}$$

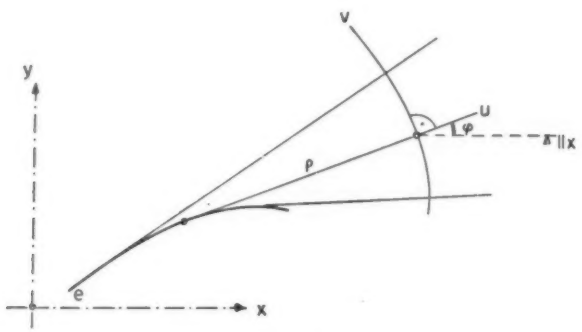

FIG. 3.

If we introduce $\rho = \rho_2$ and φ as new coordinates (Fig. 3) the first of Eqs. (1) gives

$$\frac{\partial(\rho M_1)}{\partial \rho} - M_0 = \rho Q_1 , \tag{10}$$

and the third of Eqs. (1) becomes

$$\frac{\partial(\rho Q_1)}{\partial \rho} + p\rho = 0. \tag{11}$$

This is taken from a manuscript by R. T. Shield. The integration of (10) and (11) leads to the expressions

$$M_1 = M_0 + A(\varphi) + \frac{B(\varphi)}{\rho} + \frac{1}{\rho}\int \rho^2 p \, d\rho - \int p\rho \, d\rho, \tag{12}$$

$$Q_1 = \frac{A(\varphi)}{p} - \frac{1}{\rho}\int p\rho \, d\rho, \tag{13}$$

where A and B are functions of φ only. The evolute e and the functions A and B are to be determined from the conditions of the problem, such as joining a region in regime AB to regions in other plastic regimes.

Regime BC. In this case we have $\kappa_1 = -\kappa_2$ which is the definition of a minimal surface. On such a surface the lines of principal curvature form an *isometric net* (Weatherburn [8]) and isometric coordinates u, v may be introduced for which $E = G$. If x, y are rectangular Cartesian coordinates, then $z = x + iy$ is an *analytic function* of $\chi = u + iv$ (see, for example [7] p. 179). Also $\varphi = \arg(dz/d\chi) = \arg z' = \operatorname{Im}(\log z')$ is a harmonic function (Fig. 4).

The equilibrium equation (6) becomes with $E = G$ and $\delta = -M_0/2$ (from the yield condition)

$$\Delta\omega - 2M_0\varphi_{uv} = -pE, \tag{14}$$

where Δ is the Laplace operator referred to u, v. As $E = x_u^2 + y_u^2 = |z'|^2$, and as φ_{uv} is the real part of the analytic function

$$(\log z')' = \frac{z'z''' - (z'')^2}{(z')^2},$$

Eq. (14) may be written

$$\Delta\omega - 2M_0 \operatorname{Re}\left[\frac{z'z''' - (z'')^2}{(z')^2}\right] = -p\,|z'|^2. \tag{15}$$

We remark that instead of φ the conjugated function ψ may be used. Equation (14) may be easily transformed to the principal shear-lines, which also form an isometric net.

For the special case $p = 0$ one can find immediately two particular solutions of (14),

$$\begin{aligned} \omega_1 &= u\varphi_v M_0 = uM_0 \cdot \operatorname{Re}\left[\frac{z''}{z'}\right], \\ \omega_2 &= v\varphi_u M_0 = vM_0 \cdot \operatorname{Im}\left[\frac{z''}{z'}\right]. \end{aligned} \tag{16}$$

The general solution is therefore

$$\omega = \omega_1 + \omega_0 , \quad \text{or} \quad \omega = \omega_2 + \omega_0 , \tag{17}$$

where ω_0 is any harmonic function.

4. Boundary conditions. We suppose that the plate is at yield in bending near the boundary and discuss the three following types of boundary conditions: *a*) simply supported edge, *b*) clamped edge, *c*) free edge.

a) *The simply supported edge.* For the velocity field we have $W = 0$ at the edge, where W denotes the rate of normal deflection of the plate. For the moment field the moment M_n , acting in the direction of the normal n to the boundary, vanishes. This shows that at the edge either the two principal moments have opposite signs, or at least one of them is zero. We are therefore in regime B when the boundary is a principal stress trajectory, or in regime BC when the trajectories meet the boundary at an angle. In the latter case $\omega = \frac{1}{2}M_0 \cos 2\alpha$, α being the angle between the direction 1 and the outwards-drawn normal.

b) *Clamped edge.* Along a clamped edge where deformation of the plate occurs the edge is a principal stress trajectory.

To prove this let us suppose first that no hinge occurs along the edge. If u, v are orthogonal parameters such that at the edge $u = 0$, and if

$$\mathrm{II} = L\,du^2 + 2M\,du\,dv + N\,dv^2 \tag{18}$$

is the second fundamental form of the surface into which the plate is deforming, we conclude from $W_u = W_v = W_{vv} = 0$ along the boundary $u = 0$, that

$$M = N = 0. \tag{19}$$

This shows that the edge is a line of principal curvature and therefore a line of principal stress. In the case $L = 0$ it follows from the fact that as $u \to 0$, $N/L \to 0$. If a hinge occurs, then $| M_n | = M_0$ is necessarily a principal moment. If the plate is bent downwards, then in the latter case $\omega = -M_0/2$.

c) *Free edge.* As for the simply supported edge we have the condition $\omega = (M_0/2) \cos 2\alpha$. In addition the shearing force vanishes at the edge,

$$\left(Q_n - \frac{\partial M_{nt}}{\partial t}\right)_{\text{edge}} = 0,$$

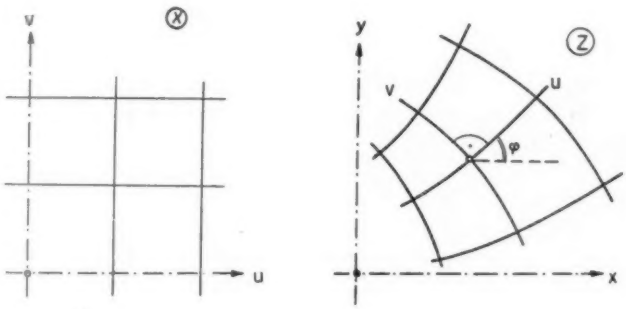

FIG. 4.

or

$$Q_n = -\frac{M_0}{2}\frac{\partial}{\partial t}(\sin 2\alpha) = -2\omega\frac{\partial \alpha}{\partial t}, \tag{20}$$

where Q_n can be calculated from (4), and where $\partial/\partial t$ denotes differentiation along the edge.

5. Discontinuities. The question of discontinuities in the moment field has been discussed previously by Prager [9] for the von Mises yield condition. In the following we investigate a discontinuity separating two regions which may or may not be in the same plastic regime. Figure 5 shows the plane of Mohr's circles and the image of Tresca's yield condition which is an oval of diameters M_0 and $2M_0$. Looking at the possible intersection of two circles, we see that three essentially different possibilities of strong discontinuities can arise, namely a jump of type "$BC \rightarrow BC$", which can be $BC \rightarrow BC$, $BC \rightarrow EF$, $EF \rightarrow EF$ (see notation of Fig. 1), or a jump of type "$BC \rightarrow AB$", or finally a jump of type "$BC \rightarrow B$".

a) Jump of type "$BC \rightarrow BC$". Denoting by M_n , M_{nt} the moments acting on an element of the line of discontinuity and by α_a , α_i the angles of the stress trajectories on both sides with respect to the normal, we conclude from Fig. 5 that

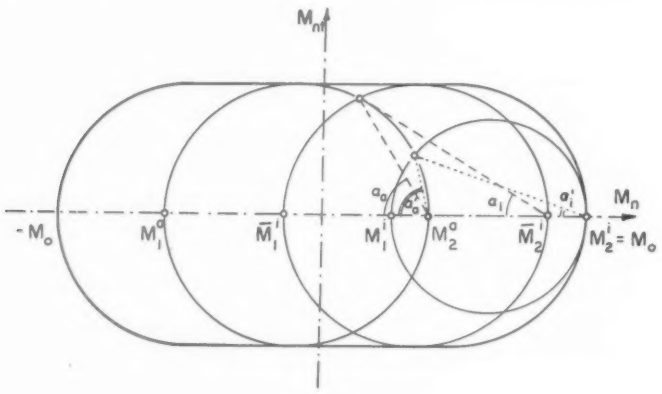

FIG. 5.

$$\alpha_i = \frac{\pi}{2} - \alpha_a \text{ (or } \alpha_i = \pi - \alpha_a), \tag{21}$$

because M_n , M_{nt} are continuous. When there is deformation of the plate near the discontinuity and assuming that $\alpha_a \neq 0, \pi/2$, both of the curvature-rates κ_t and κ_{nt} must be continuous. This requires

$$\left.\begin{aligned} \kappa_1^i \cos 2\alpha_i &= \kappa_1^a \cos 2\alpha_a , \\ \kappa_1^i \sin 2\alpha_i &= \kappa_1^a \sin 2\alpha_a , \end{aligned}\right\} \tag{22}$$

because $\kappa_1 = -\kappa_2$ on both sides of the line. From this and from (21) it follows that $\kappa_1 = \kappa_2 = 0$ on the line of discontinuity which would therefore be a rigid fiber and a line of principal curvature. Thus α_i , $\alpha_a = 0, \pi/2$, but then κ_1 , κ_2 can have any value.

A moment discontinuity between two regions which are both in the plastic regime BC is not permissible, but can take place between BC and EF.

b) Jump of type "$BC \rightarrow AB$". It is sufficient to consider the two cases of a jump from BC to AB and the case of a jump from BC to AF, all other cases being similar. Let us start with the first one.

The continuity of M_n and M_{nt} requires

$$\left.\begin{aligned} M_1^i \cos^2 \alpha_i' + M_0 \sin^2 \alpha_i' &= M_1^a + M_0 \sin^2 \alpha_a' , \\ (M_0 - M_1^i) \sin 2\alpha_i' &= M_0 \sin 2\alpha_a' , \end{aligned}\right\} \tag{23}$$

index i belonging to AB, index a to BC. The velocity field satisfies the conditions

$$\left.\begin{aligned} \kappa_2^i \cos^2 \alpha_i' &= \kappa_2^a \cos 2\alpha_a' , \\ \kappa_2^i \sin 2\alpha_i' &= 2\kappa_2^a \sin 2\alpha_a' . \end{aligned}\right\} \tag{24}$$

From (24) we get, κ_2^i , κ_2^a supposed non-zero,

$$\left.\begin{aligned} \operatorname{tg} 2\alpha_a' &= \operatorname{tg} \alpha_i' , \\ 2\alpha_a' &= \alpha_i' + m\pi , \end{aligned}\right\} \tag{25}$$

where m is 0 or 1. Equations (23) are now

$$\left.\begin{aligned} M_1^i \cos^2 \alpha_i' + M_0 \sin^2 \alpha_i' &= M_1^a + \frac{M_0}{2} (1 \mp \cos \alpha_i'), \\ 2(M_0 - M_1^i) \cos \alpha_i' &= \pm M_0 , \end{aligned}\right\} \tag{26}$$

where the upper sign refers to the case $m = 0$. Taking either sign we get from (26) $M_1^a = M_0/2$, which is not true. The assumption κ_2^i , $\kappa_2^a \neq 0$, is therefore false. Again we conclude that α_i' , $\alpha_a' = 0, \pi/2$. Similar considerations for the angles can be made for jumps between BC and AF.

From regime BC we pass to AB over a line of principal curvature which is in regime B. If this line is not straight it must be a line of the set 2 of the stress trajectories and $M_n = M_1 = 0$. The discontinuity is then a weak discontinuity in the terminology of

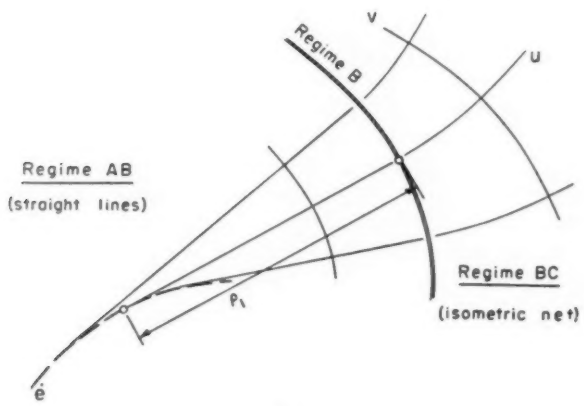

FIG. 6.

Prager [9]. Figure 6 shows the behavior of the fields when we suppose that regime B does not hold throughout a finite region.

The evolute e is connected with the regime-B-line by the condition

$$\frac{1}{\rho_1} = \frac{E_u}{2E^{3/2}}, \tag{27}$$

where E is the scale factor of the isometric net at the regime-B-line.

For the moments we have the conditions

$$M_0 + A + \frac{B}{\rho_1} + \frac{1}{\rho_1}\int \rho_1^2 p\, d\rho_1 - \int \rho_1 p\, d\rho_1 = 0, \qquad \omega = \frac{M_0}{2}, \tag{28}$$

and for the normal component of the shear-force to be continuous,

$$\frac{1}{\rho_1}\left[A - \int p\rho_1\, d\rho_1\right] = \frac{\omega_u^a}{E^{1/2}} - \frac{M_0 E_u}{2E^{3/2}}. \tag{29}$$

Until now, no non-trivial examples of such a weak discontinuity have been found.

Further the jumps of type "$BC - B$" have not yet been studied.

6. Concentrated loads. The following consideration is not a direct application, but shows how useful the discussed fields are for the determination of the limit load. Let us take a plate with a convex boundary which is *simply supported* at its edge and which is loaded by a concentrated force (Fig. 7).

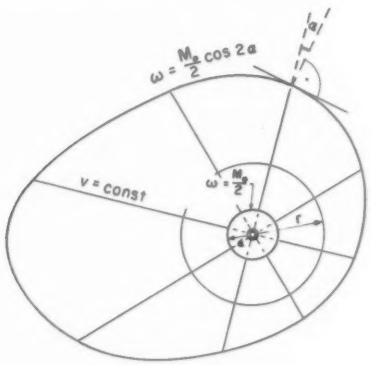

FIG. 7.

a) In order to get a *statically admissible stress field* [3] let us cut out a small circular region of radius ϵ around the force and consider the isometric net indicated in the figure. The corresponding isometric coordinates u, v are chosen such that $v = \varphi =$ const. are straight lines, and $u = \log r$, r being a measure of distance from the point of application of the load P. At the simply supported edge $\omega = \frac{1}{2}M_0 \cos 2\alpha$, and at the small circle $\omega = \frac{1}{2}M_0$. If we assume that the plate is in regime BC then Eq. (14) becomes

$$\Delta\omega = 0. \tag{30}$$

As $-M_0/2 \leq \omega \leq M_0/2$ on both boundaries, we conclude from a principal theorem on harmonic functions (Courant and Hilbert [10]) that ω satisfies the inequality

$$-\frac{M_0}{2} \leq \omega \leq \frac{M_0}{2} \tag{31}$$

in the region between the boundary and the small circle. The stress point therefore lies between the two points B, C on the yield curve, and as the harmonic function ω exists we have found a statically admissible stress field.

Let us now calculate the shearing force across the small circle. From (4) we get

$$E^{1/2} Q_1 = \omega_u - \varphi_v M_0 , \quad \varphi_v = 1, \quad -P = \int_0^{2\pi} E^{1/2} Q_1 \, dv = \int_0^{2\pi} \omega_u \, dv - 2\pi M_0 . \tag{32}$$

As ω_u is of the order $1/\log \epsilon$ the limit of this equation, when $\epsilon \to 0$, will be

$$\underline{P = 2\pi M_0 ,} \tag{33}$$

which is a lower bound for the limit load.

b) In order to find a *kinematically admissible* velocity field we take the isometric net of Fig. 8 which is given by the analytic function

$$z = f(e^{\chi}). \tag{34}$$

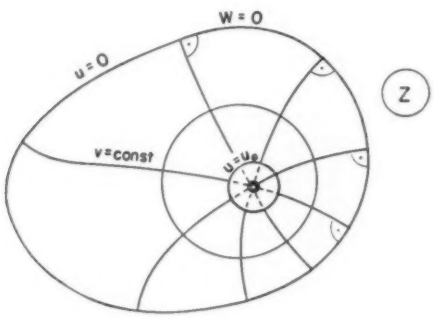

FIG. 8.

The function f is such that the transformation $\zeta = f^{-1}(z)$ maps the boundary of the plate into the unit circle and the point of application of the load into the origin $\zeta = 0$. The parameter u is then zero on the boundary and negative inside, and u has a logarithmic singularity at the loaded point. The velocity field $W = -uW_0$ satisfies the boundary conditions and the conditions of regime BC and is therefore a kinematically admissible velocity field. As the rate of work done by the concentrated load would be infinite, we replace the area inside the small "circle" $u = u_0 = \text{const.}$ by a flat area, for which the rate of dissipation is zero. It remains to calculate the dissipation outside this "circle" and at the hinge on $u = u_0$. As

$$\kappa_1 = \frac{-(L^2 + M^2)^{1/2}}{E} = \frac{-W_0(\varphi_v^2 + \varphi_u^2)^{1/2}}{E} ,$$

(this net does not represent the lines of curvature),

$$(D_{\text{total}})_{\text{outside}} = M_0 \cdot W_0 \int_0^{2\pi} dv \int_{u_0}^{0} (\varphi_u^2 + \varphi_v^2)^{1/2}\, du$$
$$\leq 2\pi M_0 W_0 \mid u_0 \mid + M_0 W_0 \int_0^{2\pi} dv \int_{u_0}^{0} \mid \varphi_u \mid du. \tag{35}$$

At the hinge Fig. 9 shows that

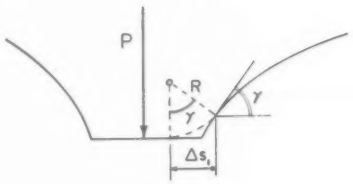

FIG. 9.

$$\operatorname{tg}\gamma = \frac{-dW}{ds_1} = \frac{-W_u}{E^{1/2}} = \frac{W_0}{E^{1/2}},$$

$$R = \frac{\Delta s_1}{\operatorname{tg}\gamma} = \frac{\Delta s_1 E^{1/2}}{W_0},$$

$$(D_{\text{total}})_{\text{hinge}} = M_0 \frac{W_0 \Delta s_1}{\Delta s_1 E^{1/2}} \cdot 2\pi E^{1/2} = 2\pi M_0 W_0 . \tag{36}$$

The second fundamental theorem of limit analysis gives therefore

$$PW_0 \leq 2\pi M_0 W_0 + \frac{2\pi M_0 W_0}{\mid u_0 \mid} + \frac{M_0 W_0}{\mid u_0 \mid} \int_0^{2\pi} dv \int_{u_0}^{0} \mid \varphi_u \mid du, \tag{37}$$

and in the limit

$$\underline{P \leq 2\pi M_0 ,} \tag{38}$$

which is an upper bound on the limit load. The exact value of the limit load is therefore $2\pi M_0$.

We remark that this result is also true for a concentrated load on a clamped plate. The same statically admissible stress field can be used, and for the velocity field we must only take into account the dissipation at the hinge produced at the clamped edge. This gives in Eq. (37) an additional term, which is of order $1/\mid u_0 \mid$ and which is zero in the limit. However as R. M. Haythornthwaite pointed out, the statement can be proved in this case in a more elementary way.

7. Saint-Venant's principle in plasticity. The application of the concentrated load is only of theoretical value, because in practice the load will always be distributed over a small area. However the examples of circular plates given by Prager and Hopkins show that the following conjecture may be true:

If the diameter of the area, where the load acts, is small in comparison with the distance from the edge, the limit load will be near the value $2\pi M_0$, and $2\pi M_0$ is the lowest limit load for the plate.

Bibliography

1. H. G. Hopkins and W. Prager, *The load carrying capacities of circular plates*, *J. Mech. Phys. Solids* **2** 1-13 (1953)
2. H. G. Hopkins, *The theory of deformation of non-hardening rigid-plastic plates under transverse load*, IUTAM Colloquium, Madrid, 176-183 (1955)
3. D. C. Drucker, W. Prager and H. J. Greenberg, *Extended limit design theorems of continuous media*, *Quart. Appl. Math.* **9**, 381-389 (1952)
4. A. R. Rzhanitsyn, *The shape at collapse of elastic-plastic plates, simply supported along the edges* (in Russian), Translated by R. M. Haythornthwaite, Tech. Rept. No. 19, Contract Nonr 562(10), Office of Naval Research
5. W. Prager, *Probleme der Plastizitätstheorie*, Birkhäuser, Basel 1955, p. 54
6. W. Prager and P. G. Hodge, Jr., *Theorie ideal plastischer Körper*, Wien, 1954, p. 133
7. W. Blaschke, *Differentialgeometrie*, Bd. 1, Berlin, 1930, pp. 112, 179
8. C. E. Weatherburn, *Differential geometry*, vol. 1, Cambridge, 1927, p. 177
9. W. Prager, *Discontinuous fields of plastic stress and flow*, Proc. Sec. Natl. Congr. Appl. Mech., June 1954
10. R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, Bd. 2, Berlin, 1932, p. 233

BOOK REVIEWS

(*Continued from p. 46*)

Matrix calculus. By E. Bodewig. North-Holland Publishing Company, Amsterdam; Interscience Publishers, Inc., New York, 1956. xi + 334 pp. $7.50.

The author feels that the usual matrix notation does not exhibit "the true building blocks of a matrix, the rows and columns", and therefore causes "discrepancy between thought and calculation, and lack of elegance". To remedy this situation, the author introduces the following notations: $A_{i.}$ = i-th row, and $A_{.j}$ = j-th column of matrix A; e_i = column vector whose i-th element is unity while all others vanish; E_{ij} = matrix having the element at the intersection of row i and column j equal to unity and all other elements equal to zero. Thus, $A_{.i} = Ae_i$ and $A_{i.} = e_i'A$, where the prime indicates the transpose; $a_{ij} = e_i'Ae_j$. These notations enable the author to represent operations explicitly that otherwise would have to be described verbally. For instance, $A(I + cE_{ij})$ is the operation that adds the c-fold of the i-th column of A to the j-th column and leaves all other columns unchanged.

The book consists of four parts, the first of which provides an excellent survey of the general theory (85 pp.). The remaining parts are concerned with specific problems of computation, each part containing sections on direct and iterative methods. Part II (96 pp.) is devoted to Linear Equations, Part III (48 pp.) to Inversion of Matrices, and Part III (98 pp. to Eigenproblems).

While often highly concise, the exposition is always clear and easy to follow, though the English is occasionally awkward. Examples of a new approach to a familiar problem abound throughout the book. In this connection, Section III, C on Geodetic Matrices merits particular mention. The author's frequent remarks on the suitability of numerical methods for different types of computing equipment will be welcomed by many readers.

W. PRAGER

Nuclear reactor physics. By Raymond L. Murray. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1957. xi + 317 pp. $10.00.

This book is a very clearly written text on the fundamental features of reactors with emphasis on neutron physics and neutron flux distribution, on transient effects in reactors, and on the temperature effects on reactor operation. Included are chapters on reactor control, transport theory, and the slowing down of neutrons.

The author has included sufficient mathematical treatment to make the discussion useful and interesting but not so much that only the specialist would be interested. The book seems to have excellent balance between what a student of reactor fundamentals might need and what a well trained physicist or mathematician might want to know without becoming a specialist. The book seems excellent for those who simply want to know what problems enter into reactor design.

ROHN TRUELL

The International Dictionary of Physics and Electronics. D. Van Nostrand Co., Inc., Toronto, New York, London, 1956. xvii + 1004 pp. $20.00.

This book is an ambitious undertaking (even for fifteen editors) amounting to one thousand pages of laws, equations, definitions, principles and concepts of physics. The objective as stated by the editors has been to provide a book useful as a general reference in physics. This work will undoubtedly be useful to many people especially to those who seek information outside their field of specialization.

ROHN TRUELL

(*Continued on p. 94*)

ON FINITE EXPANSION OF A HOLE IN A THIN INFINITE PLATE*

BY

P. G. HODGE, JR.† AND R. SANKARANARAYANAN

Polytechnic Institute of Brooklyn

Summary. A complete solution for all values of applied pressure is obtained for the expansion of a hole of zero radius in an initially uniform infinite sheet. The analysis is compared with those of previous investigators and found to be simpler and/or more complete. It is shown that the results are applicable to the expansion of a finite hole in a uniform sheet and to the expansion of a hole in a tapered sheet.

1. Introduction. The problem of finite expansion of a circular hole in a thin infinite plate has been discussed repeatedly in the literature. Taylor [3][1] treated the problem by using Tresca's yield condition and the flow rule associated with Mises' yield condition, but he considered a rigid perfectly plastic material. Hill's [4] extension of Taylor's work was also restricted to a non-strain hardening material. Prager [1] solved the problem for a rigid strain hardening material which satisfies Tresca's initial yield condition and the associated flow rule, but his solution is restricted to a certain finite value of the pressure. Alexander and Ford [5] have treated the same problem, using Mises' yield condition and the associated Prandtl-Reuss relations.

The present paper deals with the same problem treated by Prager, but the solution is obtained for all values of the internal pressure. As in Hill's treatment [4] the mathematical analysis in the plastic regions is simplified by reducing the problem to the solution of ordinary differential equations by means of a substitution of independent variables.

Specifically, a hole is expanded from zero radius in a thin infinite sheet of work-hardening material which satisfies Tresca's yield condition and the associated flow rule. An incompressible material is envisaged, and since finite plastic deformations are considered, elastic deformations are neglected.

In the case of plane stress problems Tresca's yield condition can be represented by a hexagon in the σ_1 , σ_2 plane (Fig. 1). If the "stress point" with coordinates σ_1 , σ_2 remains in the interior of the hexagon, the material is rigid. For a point on the boundary, plastic flow may commence in accord with the plastic potential flow law [6]. According to this law, on a side of the hexagon the strain rate vector must be the unique normal, while at a corner it may assume any position between the limiting normals.

For a perfectly plastic material the hexagon in Fig. 1 is fixed, and the stress point is not permitted to leave it. For the isotropic strain hardening material considered in this paper, the yield curve maintains its shape, center and orientation, but merely changes its size.

It has been shown [7, 8] that under certain restrictive conditions the plastic flow laws can be explicitly integrated. These conditions, defined as a "regular progression"

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†Now at Illinois Institute of Technology.

[1]Numbers in square brackets refer to the references collected at the end of the paper.

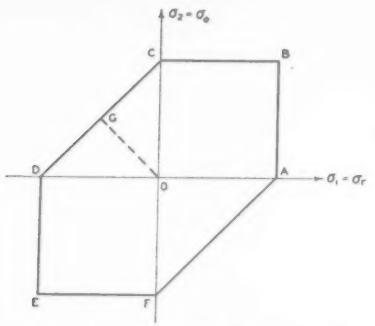

FIG. 1. Tresca hexagon for plane stress.

of the stress point, state that it must never lose contact with a side of the hexagon. In other words, the stress point may stay on a side, stay in a corner, or move from a side to a corner; it may not move from one side to another, from a corner to a side, or back into the interior.

Prager [1] considered the enlargement of a circular hole of radius a_0 to $1.48a_0$ by the application of a gradually increasing uniform pressure p to the edge of the hole. The stress point of any particle starts at the origin for $p = 0$ and moves along the straight line OG as p increases (Fig. 2). It first becomes plastic at G and moves along GD to D. Next, the stress point remains a finite time at D, engaging in plastic flow. Finally, it moves back along DC to the accompaniment of further plastic flow. However, Prager did not consider the final stage of plastic flow, when "non-regular progression" of the stress point takes place from the corner D, back to the side CD of the yield hexagon. Therefore, as he pointed out, his solution is only valid for $a/a_0 < 1.48$. In particular, Prager's solution is not applicable to the expansion of a hole from zero radius.

Since the stress point progresses non-regularly in the final stage of plastic deformation, the explicit integrated forms of the flow law cannot be used. A similar condition was encountered in an earlier paper [2] concerned with the bending of a simply supported annular plate. There it was found that the complete solution was built up of zones and that appropriate initial conditions were always available from plastic zones already formed. The analysis of the present paper reveals that a similar condition holds in the finite expansion of a hole in an infinite sheet.

For conceptual simplicity, it is first assumed that the initial radius of the hole is zero. Since the plate is infinite, this implies that there is no characteristic length in the plane of the sheet. It follows that the geometric solution at all times must be similar to itself. Therefore, rather than the solution depending upon time and space independently, it depends only upon a conveniently chosen space-time ratio. This technique was first used by Hill [4].

The solution for a hole expanded from a finite radius is obtained from that for expanding a hole from zero radius simply by discarding the part of the solution which is not required. This is so because it is immaterial whether the pressure at any radius is applied by an external agency or through the displacement of an inner annulus of the sheet. The stress and velocity in an element depend only on what happens beyond the radius.

The basic theory is presented in the next section. Section 3 gives the details of the solution. The paper will conclude with a discussion of some limitations and extensions of the analysis, as well as its relation to other investigations cited.

2. Basic theory. The state of a thin infinite sheet of initially uniform thickness is fully defined by the principal stresses σ_r and σ_ϑ, the radial displacement u, and the thickness h. The stresses must satisfy the equation of radial equilibrium

$$\frac{\partial(h\sigma_r)}{\partial r} + \frac{h(\sigma_r - \sigma_\vartheta)}{r} = 0. \tag{1}$$

Where the material is rigid, it may be regarded as the limiting case of an elastic material as Young's modulus becomes infinite. Therefore, in a rigid region the stresses must also satisfy the compatibility equation for an incompressible material

$$r\left(\frac{\partial\sigma_\vartheta}{\partial r} - \frac{1}{2}\frac{\partial\sigma_r}{\partial r}\right) + \frac{3}{2}(\sigma_\vartheta - \sigma_r) = 0. \tag{2}$$

A stress point in the interior of the hexagon must satisfy Eqs. (1) and (2). The flow law in a plastic region is expressed in terms of the plastic strain rates

$$\epsilon_r = \partial v/\partial r, \qquad \epsilon_\vartheta = v/r, \qquad \epsilon_z = (1/h)(\partial h/\partial t), \tag{3}$$

where v is the radial velocity. Since the material is incompressible,

$$\partial v/\partial r + v/r + \partial(\log h)/\partial t = 0. \tag{4}$$

The plastic flow law assumes different forms for each side or corner of the yield hexagon (Fig. 1). For the type of loading considered here, the relevant plastic regimes are CD and D. On CD, the strain rate vector must be directed normal to the side, hence

$$\epsilon_r + \epsilon_\vartheta = -\epsilon_z = 0. \tag{5}$$

Further, the stresses must satisfy

$$\sigma_r \leq 0, \qquad \sigma_\vartheta \geq 0 \tag{6}$$

if the stress point is to remain on CD.

In the corner D, the stress point must always lie on the σ_r axis; hence

$$\sigma_\vartheta = 0. \tag{7}$$

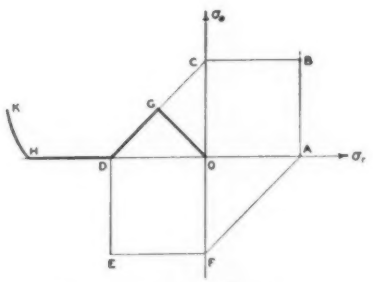

FIG. 2. The stress profile.

The strain rate vector must lie between the values corresponding to the sides CD and DE; hence

$$0 < \epsilon_\vartheta < -\epsilon_r \,. \tag{8}$$

It remains to consider the rate of hardening. As in Prager's treatment [1] this will be assumed to be proportional to the energy dissipation. Thus

$$\partial\sigma/\partial t = \alpha D, \tag{9}$$

where α is a constant, D the dissipation function, and

$$\sigma = \max [|\, \sigma_r \,|, \quad |\, \sigma_\vartheta \,|, \quad |\, \sigma_r - \sigma_\vartheta \,|]. \tag{10}$$

On the side CD, as well as the corner D, Eq. (9) can be written

$$\partial\sigma/\partial t = -\alpha\sigma\epsilon_r \,,$$

where

$$\sigma = \sigma_\vartheta - \sigma_r \,. \tag{11}$$

At $r = \infty$, the radial stress must vanish, and at $r = a$, the current hole radius, it must be in equilibrium with the applied load. Thus

$$\text{at} \quad r = a, \qquad h\sigma_r = -p; \tag{12}$$

$$\text{at} \quad r = \infty, \qquad \sigma_r = 0. \tag{13}$$

Finally, at the beginning of the expansion the thickness must be equal to its initial constant value

$$\text{at} \quad t = 0, \qquad h = h_0 \,.$$

3. Solution. When a hole is expanded from zero radius, the material at a sufficiently large distance from the hole is rigid because it is stressed below the yield limit. Therefore, $h = h_0$ and $u = 0$. At $r = \infty$, the stresses must vanish, while at the elastic-plastic boundary they must satisfy

$$\sigma_r - \sigma_\vartheta = -\sigma_0 \,. \tag{14}$$

Therefore, Eqs. (1) and (2) determine the stresses to be

$$\sigma_r = -\sigma_\vartheta = \frac{-\sigma_0}{2}\frac{\rho^2}{r^2} \,, \tag{15}$$

where σ_0 denotes the yield stress of the virgin material and ρ is the radius of the elastic plastic interface. The solution given by Eqs. (15) is valid for $r > \rho$.

For r somewhat less than ρ, the stress point will be in regime CD. For this portion Eq. (5) shows that h is still equal to h_0 . Since there is no thickening of the material, there can be no straining and hence it is still rigid, even though plastic. The stresses are now given by Eqs. (1) and (14), with Eq. (15) as a boundary condition at $r = \rho$. Thus

$$\begin{aligned} \sigma_r &= -\sigma_0\left(\frac{1}{2} + \log\frac{\rho}{r}\right); \\ \sigma_\vartheta &= \sigma_0\left(\frac{1}{2} - \log\frac{\rho}{r}\right). \end{aligned} \tag{16}$$

It is clear from the second Eq. (16) that the hoop stress vanishes for

$$r = \rho e^{-1/2}. \tag{17}$$

Therefore, it follows from inequalities (6) that Eqs. (16) are valid for $\mu < r < \rho$, where ρ and μ are related by

$$\rho = \mu e^{1/2}. \tag{18}$$

The portion of the sheet just inside the circle $r = \mu$ corresponds to the regime D. It is evident that μ will increase with the pressure, and hence it may be conveniently taken as the time parameter. The ratio of the original radius r_0 of a particle to the radius of this boundary is then taken as the space-time ratio mentioned in the Introduction. Thus the entire solution will be a function of

$$x = r_0/\mu. \tag{19a}$$

The equilibrium equation, the incompressibility relation, the law of strain hardening, and the flow rule may now be expressed in terms of this parameter x. The resulting equations will be ordinary differential equations, and the solutions will be obtained relatively easily. It is convenient to define the following dimensionless quantities

$$\xi = r/\mu, \qquad s_r = \sigma_r/\sigma_0, \qquad s_\vartheta = \sigma_\vartheta/\sigma_0, \qquad s = \sigma/\sigma_0, \qquad \eta = h/h_0, \tag{19b}$$

all as function of x only. If φ is any function of x alone, then the following relations are seen to hold:

$$\begin{aligned} \partial\varphi/\partial r_0 &= (1/\mu)\varphi', \\ \partial\varphi/\partial\mu &= -(1/\mu)x\varphi', \\ \partial r/\partial r_0 &= \frac{\mu\partial(r/\mu)}{\partial r_0} = \mu\frac{\partial\xi}{\partial r_0} = \xi', \\ \partial\varphi/\partial r &= \frac{\partial\varphi/\partial r_0}{\partial r/\partial r_0} = \varphi'/\mu\xi', \end{aligned} \tag{20}$$

where primes denote differentiation with respect to x. Therefore, the velocity and strain rates are

$$v = \partial r/\partial\mu = \xi - x\xi', \tag{21}$$

$$\epsilon_r = \partial u/\partial r = -x\xi''/\mu\xi', \tag{22}$$

$$\epsilon_\vartheta = v/r = (\xi - x\xi')/\mu\xi, \tag{23}$$

$$\epsilon_z = \frac{1}{h}\frac{\partial h}{\partial\mu} = -(x\eta')/\mu\eta. \tag{24}$$

At the boundary of the plastic region $x = 1$, the displacement and velocity must vanish, the thickness must equal its initial value, and the stress point must be at D. It follows that, at $x = 1$,

$$\xi = 1, \qquad \xi' = 1, \qquad \eta = 1, \qquad s = 1. \tag{25}$$

The strain hardening law and the incompressibility relation, Eqs. (10) and (5), respectively, may now be written as

$$-xs' = \alpha x\xi''/\xi',$$
$$-x\xi''/\xi + 1 - x\xi'/\xi - x\eta'/\eta = 0. \tag{26}$$

Integrals of Eqs. (26) which satisfy Eqs. (25) are then found in the form

$$s^{1/\alpha}\xi' = 1, \tag{27}$$
$$\xi\xi'\eta = x. \tag{28}$$

Finally, the equilibrium equation in the corner D becomes

$$(\xi\eta s_r')/\xi' = 0. \tag{29}$$

The solution of Eqs. (27), (28) and (29) satisfying the boundary conditions (25) are

$$s = -s_r = (x)^{-\alpha/(1+\alpha)}, \qquad s_\vartheta = 0,$$
$$\xi = \frac{1}{2+\alpha}\left[1 + (1+\alpha)x^{(2+\alpha)/(1+\alpha)}\right], \tag{30}$$
$$\eta = \frac{(2+\alpha)x^{\alpha/(1+\alpha)}}{1 + (1+\alpha)x^{(2+\alpha)/(1+\alpha)}}.$$

This solution will be valid so long as Inequality (8) is satisfied. It is readily verified that this is true for all $1/2 < \xi < 1$. Substitution of Eqs. (19) into Eqs. (30) yields the results previously obtained by Prager [1].

For $\xi < 1/2$, the solutions given by Eqs. (32) to (35) predict a strain rate vector which points too far up, suggesting that the portion of the plate just inside the circle $\xi = 1/2$, corresponds to plastic regime CD.

For the solution in the region $\xi \leq 1/2$, Eqs. (27) and (28) are still valid since Eqs. (10) and (5) hold in regimes D and CD. In addition Eq. (5) implies that

$$\eta = \eta_1 = 2\left[\frac{\alpha}{2(1+\alpha)}\right]^{\alpha/(2+\alpha)}. \tag{31}$$

Here, as in the following, integration constants are determined to give continuity with Eqs. (30) at $x = 1/2$.

Because η is a constant, Eq. (28) is easily solved, the result being

$$\xi = \left[\frac{x^2}{\eta_1} + \frac{1}{4(1+\alpha)}\right]^{1/2}. \tag{32}$$

With ξ known, Eq. (27) furnishes

$$s = (\eta_1/x)^\alpha\left[\frac{x^2}{\eta_1} + \frac{1}{4(1+\alpha)}\right]^{\alpha/2}. \tag{33}$$

Hence, from the equilibrium equation

$$s_r = \int_{x_1}^{x} (x/\eta_1)^{1-\alpha}\left[\frac{x^2}{\eta_1} + \frac{1}{4(1+\alpha)}\right]^{(\alpha-2)/2} dx - \left[\frac{\alpha}{2(1+\alpha)}\right]^{-\alpha/(2+\alpha)}. \tag{34}$$

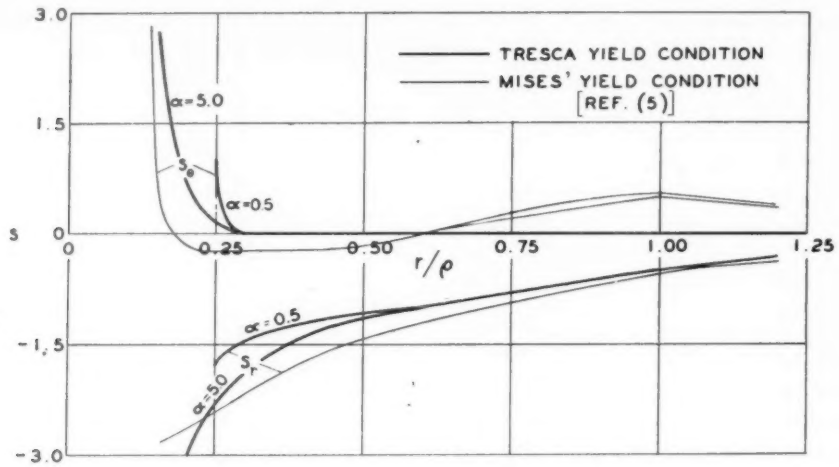

FIG. 3. Distribution of radial and circumferential stresses.

Finally, since $s_\vartheta = s + s_r$, the value of s_ϑ is obtained by adding Eqs. (33) and (34).

It is easily verified that Inequalities (6) are always valid for $0 \leq \xi < 1/2$. Therefore, the solutions represented by Eqs. (33) and (34) give the complete stress distribution in the plastic region. It is further verified that s increases as the value of ξ is decreased from 0.5, so that the strain hardening continues.

4. Conclusions. The complete solution has been obtained for the finite expansion of a circular hole from zero radius in a thin infinite plate of initially uniform thickness. The material of the plate is rigid-plastic, satisfies Tresca's yield condition and the associated flow rule, and hardens isotropically.

As already stated, the solution for a hole expanded from a finite radius is obtained from the present analysis simply by discarding the part of the solution around the edge of the hole which is not required. This is a consequence of the fact that the outer annular

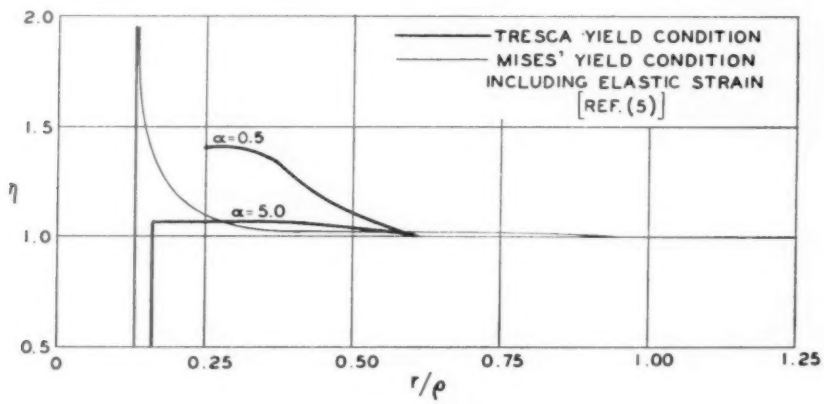

FIG. 4. Thickness variation in deformed sheet.

fibres of the plate receive radial pressure transmitted only from the inner fibres, and the inner fibres may be equally well replaced by some externally applied pressure.

As pointed out by Alexander and Ford [5], the governing equations in the plastic region of a plate of initially uniform thickness are identical with those of a plate which varies in thickness proportionately with the radius. Therefore, the plastic region analysis of the paper applies equally well to such a plate.

The great simplicity achieved by using Tresca's yield condition and the associated flow rule will be obvious by a comparison of the present analysis with that of Alexander and Ford [5], who treated the same problem by using Mises' yield condition and the associated Prandtl-Reuss relations. Therefore, it is of some interest to compare the results of the two theories. Figures (3) and (4) show the stress and thickness distributions, plotted as functions of r/ρ, for two values $\alpha = 5.0$ and $\alpha = 0.5$, of the hardening.

Figures (3) and (4) show that there is fairly good agreement between the two distributions of stress, though there is some discrepancy between the two distributions of strain. Experiment alone, of course, can decide which analysis predicts the real behavior of the material more closely. However, even if experimental evidence were to show the Mises-type solution to be more accurate, the present treatment might still be valuable as an approximate solution due to its much greater simplicity.

References

1. W. Prager, *On the use of singular yield conditions and associated flow rules*, J. Appl. Mech. **20**, 317-320 (1953)
2. P. G. Hodge, Jr., *Plastic bending of an annular plate*, J. Math. & Phys. **36**, 130–137 (1957)
3. G. I. Taylor, *The formation and enlargement of a circular hole in a thin plastic sheet*, Quart. J. Mech. Appl. Math. **1**, 103-124 (1948)
4. R. Hill, *Plastic distortion of non-uniform sheets*, Phil. Mag. Ser. 7, vol. **xi**, 971-983 (1949)
5. J. M. Alexander and H. Ford, *On expanding a hole from zero radius in a thin infinite plate*, Proc. Roy. Soc. **A226**, 543-561 (1954)
6. R. v. Mises, *Mechanik der plastischen Formaenderung von Kristallen*, ZAMM **8**, 161-185 (1928)
7. W. Prager, *The theory of plasticity—a survey of recent achievements*, James Clayton Lecture, Inst. Mech. Engrs., London, 1955
8. P. G. Hodge, Jr., *Minimum principles of piecewise linear isotropic plasticity*, J. Ratl. Mech. and Anal. **5**, 917-938 (1956)

—NOTES—

CROCCO'S VORTICITY LAW IN A NON-UNIFORM MATERIAL*

By C. S. WU and W. D. HAYES (*Princeton University*)

In the derivation of Crocco's vorticity law [1] there is a basic restriction, that the material in the flow field be uniform, that the flow be isocompositional. This restriction is not generally appreciated, and is only implied in most published derivations of this law. The purpose of this note is to express the corresponding form of the vorticity law valid with non-uniform material, such as a gas mixture with non-zero concentration gradients or a material in which a chemical reaction is proceeding.

The basic assumptions which are needed are: (i) no body forces except gravity; (ii) no frictional stresses; (iii) steady flow; (iv) thermodynamic quasi-equilibrium; and (v) continuous first derivatives of thermodynamic variables and velocity. With assumption (iv) the material obeys a differential state equation of the form

$$dh = T\,ds + \frac{dp}{\rho} + \sum \mu_k\,dn_k\,, \tag{1}$$

where μ_k and n_k are the molar chemical potential and the specific molar concentration, respectively, of the kth constituent, p and T are the pressure and absolute temperature, and h, s, and $1/\rho$ are the specific enthalpy, entropy and volume.

Since the material everywhere in the flow field obeys Eq. (1), we may write the corresponding gradient equation

$$\nabla h = T\nabla s + \frac{\nabla p}{\rho} + \sum \mu_k \nabla n_k\,. \tag{2}$$

With assumptions (i), (ii) and (iii) we may write the equation of motion of the fluid

$$\nabla \frac{q^2}{2} - \mathbf{q} \times (\nabla \times \mathbf{q}) + \frac{\nabla p}{\rho} + \nabla\Omega = 0, \tag{3}$$

where Ω is the gravitational potential. The total enthalpy h_0 is defined

$$h_0 = h + \frac{q^2}{2} + \Omega. \tag{4}$$

Eqs. (2), (3), and (4) may be combined to give the desired vorticity law

$$\mathbf{q} \times (\nabla \times \mathbf{q}) = \nabla h_0 - T\nabla s - \sum \mu_k \nabla n_k\,. \tag{5}$$

If there is diffusion present the velocity is defined as a mass mean, as the total mass flow vector divided by the total density.

In the absence of any real-fluid effects in an isocompositional flow, both h_0 and s are constant along streamlines, and ∇h_0 and ∇s are both directed normal to $\mathbf{q}$. These real-fluid effects are viscosity, excluded by assumption (ii), heat conduction, and relaxation.

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With a non-uniform material $\sum \mu_k \, dn_k$ is zero along a streamline if there is no diffusion and if any chemical reaction is reversible. Thus, since diffusion and irreversibility of chemical reaction are real-fluid effects associated with varying composition, no new feature appears in the generalized form of Crocco's vorticity law.

The vorticity law is most useful if h_0 and the n_k's are constant over the entire flow field; in this case the entropy gradient may be obtained directly from the vorticity or the lateral components of the vorticity from the entropy gradient. However, with no real-fluid effects present and with sufficient knowledge of the flow field the vorticity law still gives a useful relation between vorticity and entropy gradient, as then h_0 is a function only of the streamline and the n_k's may in principle be obtained, and the lateral gradients of these quantities may be calculated.

In a binary mixture we may set

$$c = n_1 M_1 \, , \tag{6a}$$

$$1 - c = n_2 M_2 \, , \tag{6b}$$

$$\mu = \frac{\mu_1}{M_1} - \frac{\mu_2}{M_2} \, , \tag{6c}$$

where M_1 and M_2 are the molecular weights (molar mass) of the two components. With this notation we have

$$\sum \mu_k \, dn_k = \mu \, dc \tag{7}$$

and Crocco's vorticity law takes the form

$$\mathbf{q} \times (\nabla \times \mathbf{q}) = \nabla h_0 - T \nabla s - \mu \nabla c. \tag{8}$$

Reference

[1] L. Crocco, *Ein neue Stromfunktion für die Erforschung der Bewegung der Gase mit Rotation*, ZAMM **17**, 1-7 (1937)

THE SCHWARZIAN DERIVATIVE AND THE APPROXIMATION METHOD OF BRILLOUIN*

By AUREL WINTNER (*The Johns Hopkins University*)

1. Let the coefficient function of the differential equation

$$x'' + f^2(t)x = 0 \tag{1}$$

be the square of a positive function $f(t)$ which is given, and possesses a continuous second derivative, for large positive t, say for $t_0 \leqq t < \infty$. Then the function

$$\varphi(t) = \int_{t_0}^{t} f(s) \, ds \tag{2}$$

is strictly increasing and has a continuous third derivative, whereas the function

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$$4F(t) = 3f'^2(t)/f^4(t) - 2f''(t)/f^3(t) \tag{3}$$

exists and is continuous.

In view of (2), the definition of $F(t)$ is equivalent to

$$F(t) = -2f^2(t)\{\varphi(t)\}, \tag{4}$$

if $\{\ \}$ is the symbol of the differential parameter of Lagrange-Schwarz,

$$\{\varphi\} = (\varphi''/\varphi')' - (\varphi''/\varphi')^2/2. \tag{5}$$

It is also seen from (3) that

$$F(t)f(t) = k(t)k''(t), \text{ where } k(t) = [f(t)]^{-1/2}. \tag{6}$$

2. If $t \to \infty$, then the function (2), being increasing, tends either to a finite limit $\varphi(\infty)$ or to $\varphi(\infty) = \infty$. Suppose that the latter is the case,

$$\int^{\infty} f(t)\,dt = \infty \qquad (f > 0), \tag{7}$$

and that the function (6) is absolutely integrable,

$$\int^{\infty} |\,F(t)\,|\,f(t)\,dt < \infty \qquad (f > 0). \tag{8}$$

The following theorem was proved ten years ago[1] (and, in certain particular cases of it, was repeatedly rediscovered[2] in the meantime):

For large positive t, let $f(t)$ be a positive, twice continuously differentiable function satisfying conditions (7) *and* (8), *where $F(t)$ is defined by* (3). *Then the general solution of the differential equation* (1) *is the superposition of two linearly independent solutions which are asymptotic (as $t \to \infty$) to the real and imaginary parts of the function*

$$y(t) = [f(t)]^{-1/2} \exp\left[i\int_{t_0}^{t} f(s)\,ds\right]. \tag{9}$$

In addition, differentiation of this asymptotic representation is allowed.

In other words, the assumptions (7) and (8) are sufficient in order that the general solution of the differential equation (1) be of the form $x(t) = c_1x_1(t) + c_2x_2(t)$, where, as $t \to \infty$,

$$x_1(t) \sim y_1(t), \quad x_2(t) \sim y_2(t) \quad \text{and} \quad x_1'(t) \sim y_1'(t), \quad x_2'(t) \sim y_2'(t), \tag{10}$$

if the real functions $y_1(t)$, $y_2(t)$ are defined by placing

$$y_1(t) + iy_2(t) = [f(t)]^{-1/2} \exp[i\varphi(t)]. \tag{11}$$

3. The point in the theorem just quoted was the generality of its assumptions (and, correspondingly, the comparative simplicity of its proof), since, under assumptions

[1] A. Wintner, Phys. Rev. **72**, 516-517 (1947).

[2] See, e.g., M. Zlámal, Czechoslov. Math. J. **6** (81), 75-93 (1956) (in German, with a Russian abstract), where one of the books of R. Bellman (1953) is mentioned. The main result of Zlámal's Chap. II (Theorem 6 on p. 83, along with the proof on pp. 84-85) is the simplest of the corollaries of the note in the Phys. Rev., referred to in the preceding footnote; see the general theorem italicized above, which shows that the problem is independent of Zlámal's convexity assumption.

more stringent than the pair of conditions (7) and (8), the asymptotic form (9) was well known from the mathematical as well as from the physical literature of the subject.[3] The proof of the theorem depended on a combination of two steps: (α) the standard use of the formal substitution of the Liouville[4] and Riemann,[5] and (β) the application of an "Abelian lemma," which runs as follows.[6]

If $\epsilon = \epsilon(\varphi)$ is defined, for large positive φ, as an arbitrary continuous (possibly not even real-valued) function which is absolutely integrable,

$$\int^{\infty} | \epsilon(\varphi) | \, d\varphi < \infty , \tag{12}$$

then the general solution $z = z(\varphi)$ of the differential equation

$$d^2z/d\varphi^2 + [1 + \epsilon(\varphi)]z = 0 \tag{13}$$

(as well as the derivative, $dz(\varphi)/d\varphi$, of every solution) has, as $\varphi \to \infty$, the same asymptotic behavior as what results in the trivial case, $\epsilon(\varphi) \equiv 0$, of (13); so that

$$z(\varphi) - (c_1 \sin \varphi + c_2 \cos \varphi) \to 0, \qquad dz(\varphi)/d\varphi - (c_1 \cos \varphi - c_2 \sin \varphi) \to 0 \tag{14}$$

as $\varphi \to \infty$. In the application of this Abelian lemma, the independent variable φ must be chosen to be the function defined by (2) or (4) above. The rôle of step (α) was precisely the reduction of (1) to (13), where $\epsilon(\varphi)$ is "small" (for large φ) in an appropriate sense, whereas step (β) consisted in assuring that condition (12) is a sufficient "degree of smallness" for the asymptotic form (14) of the general solution of (13).

In what follows, a more general and less artificial approach will be taken to the asymptotic behavior of the solutions $x(t)$ of (1) and of their derivatives. The improvements will depend on a formal fact pointed out by Brillouin.[7]

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[3]For references, see G. Sansone, *Equazioni differenziali nel campo reale* vol. 2, Bologna, 1949, p. 53; see also pp. 101-110 of Titchmarsh's book, referred to in footnote[5].

[4]See G. Sansone, *op. cit.*, vol. 2, pp. 68-70.

[5]B. Riemann, *Gesammalte mathematische Werke*, 2nd ed., Leipzig, 1892, pp. 335-336, or the presentation in E. G. C. Poole's *Theory of linear differential equations*, Oxford, 1936, pp. 121-122; see also E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations*, Oxford, 1946, p. 102.

[6]A proof, based on successiive approximations, was given by H. Weyl, *Gruppentheorie und Quantenmechanik*, Leipzig, 1928, pp. 61-63. Actually, as pointed out in my note in the Phys. Rev., referred to above[1], the same proof is contained already in an important paper of M. Bôcher (1900). For another arrangement of the proof, a proof based on the "Abelian" aspect, see A. Wintner, Amer. J. Math. 69, 251-272 (1948).

As shown recently [A. Wintner, Amer. J. Math. 78, 895-897 (1956)], the latter proof is such as to lead to simple criteria which are substantially more general than (12) and are, nevertheless, sufficient in order that (14) should hold for the general solution of (13).

As pointed out at the end of the paper just mentioned, the question of asymptotic behavior is stricter than that of stability (boundedness). For an application of the Loiuville-Riemann transformation to the latter question, see U. Barbuti, Annali di Pisa, ser. 3, 8, 81-91 (1954), where, with regard to the question of stability, recent papers of G. Ascoli (1947), L. A. Gusarov (1949) and U. Barbuti (1952) are cited.

[7]L. Brillouin, Quart. Appl. Math. 6, 166-178 (1948) and 7, 363-380 (1950) (in particular p. 172 and p. 364 respectively); see also L. Brillouin and M. Parodi, *Propagation des ondes dans les milieux périodiques*, Paris, 1956, p. 216. Strictly speaking, in what Brillouin is interested is, not the asymptotic behavior, for indefinitely large t, of the solutions $x(t)$ of a fixed differential equation (1), but a situation in which the t-range is limited but the differential equation (1), instead of being fixed, contains a variable parameter.

4. For large positive t, let $f(t)$ be any, perhaps not even real-valued, twice continuously differentiable function which is nowhere zero,

$$f(t) \neq 0 \quad \text{for} \quad t_0 \leqq t < \infty \tag{15}$$

[which, if $f(t)$ is real-valued or purely imaginary, means that

$$f^2(t) > 0 \quad \text{or} \quad f^2(t) < 0 \tag{16}$$

throughout]. Then, with either of the continuous determinations of the square root occurring in it, Eq. (9) defines a twice continuously differentiable function

$$y(t) \neq 0 \quad \text{for} \quad t_0 \leqq t < \infty . \tag{17}$$

Two differentiations of the definition (9) show that

$$y''(t) + [f^2(t) - f^2(t)F(t)]y(t) = 0 \tag{18}$$

is an identity, if $F(t)$ is defined by (3) [that is, by (4) and (2) together or, equivalently, by (6)]. Starting with the identity (18), Brillouin observes that the function (9) can be expected to furnish an approximation to a solution $x(t)$ of (1) when the coefficient, [ ], of $y(t)$ in (18) is "sufficiently close" to the coefficient, $f^2(t)$, of $x(t)$ in (1), that is, if $f^2(t)F(t)$ is "sufficiently small."[8]

The issue is, of course, the specification of conditions under which such a "degree of smallness" (for large t) will actually be realized. But it turns out that such a specification can be obtained, in explicit terms, from a general criterion. The criterion in question is as follows.[9]

If $y = y_1(t)$ and $y = y_2(t)$ are two, linearly independent, solutions of a differential equation

$$y'' + g(t)y = 0 \tag{19}$$

in which $g(t)$, where $t_0 \leqq t < \infty$, is any continuous function, and if $h(t)$, where $t_0 \leqq t < \infty$, is any continuous function having the property that the corresponding differential equation

$$x'' + h(t)x = 0 \tag{20}$$

is so "close" (as $t \to \infty$) to the differential equation (19) that

$$\int^{\infty} (| y_1(t) |^2 + | y_2(t) |^2) \, | g(t) - h(t) | \, dt < \infty \tag{21}$$

holds, then the two differential equations are *asymptotically equivalent*, in the following sense. The differential equation (20) will possess two linearly independent solutions, $x = x_1(t)$ and $x = x_2(t)$, which, along with their derivatives, will satisfy the asymptotic relations (10) as $t \to \infty$.

5. In order to combine this criterion with Brillouin's method, let (18) and (1) be identified with (19) and (20) respectively (so that $| g - h |$ becomes $| f^2F |$) and, under the assumption that $f(t)$ is positive, let $y_1(t)$ and $y_2(t)$ be those (real-valued) solutions

[8] In this regard, see the comments made at the end of the preceding footnote.

[9] A. Wintner, Quart. Appl. Math. 15, 428–430 (1957).

of (18) for which $y_1(t) + iy_2(t)$ becomes the (complex-valued) function (9). Then the condition (21) reduces to the convergence of the integral

$$\int^{\infty} | y(t) |^2 f^2(t) | F(t) | dt,$$

since $| y_1 |^2 + | y_2 |^2 = | y |^2$ and $| g - h | = | F | f^2$. But the convergence of this integral is equivalent to the condition (8), since $| y |^2 f^2 = f$, by the definition, (9), of y, where $| \exp | = 1$.

Accordingly, the general criterion of Sec. 4 supplies, as a corollary, not only the theorem quoted in Sec. 2, but also an improvement of that theorem. In fact, it is now seen that *the assumption* (7) *of the theorem italicized above can be omitted*, that is, only the other assumption, (8), of the theorem is needed.

6. This reduction of the assumptions of the theorem could have been proved directly. The main point was, however, the possibility of a straightforward approach, one replacing the formalism of the Liouville-Riemann substitution by its simplified version, the formal identity of Brillouin.

The spurious character of the restriction (7) is indicated even by Brillouin's elemen tary case. In fact, the latter results by assigning the following condition for the coefficient function of (1):

$$F(t) = 0 \quad \text{for} \quad t_0 < t < \infty. \tag{22}$$

In view of the definition (3), the assignment (22) represents a differential equation of second order for $f(t)$, and Brillouin's approximation theory (*loc. cit.*) depends on the circumstance that the general solution of the (non-linear) differential equation (22) is

$$f(t) = C_1/(t + C_2)^2, \tag{23}$$

(C_1 and C_2 are the integration constants), as is readily verified by direct substitution. Clearly, (22) means that the differential equation (18) is not only "*close*" to, but is *identical* with, the "perturbed" differential equation (1) and, correspondingly, condition (8) is certainly satisfied. But condition (7) happens to be violated. In fact, if (without loss of generality) $C_1 = 1$ and $C_2 = 0$, then the general solution, (23), of (22) reduces to $f(t) = t^{-2}$, and so

$$\int_{t_0}^{\infty} f(t)\, dt = \int_{t_0}^{\infty} t^{-2}\, dt < \infty \qquad (t_0 > 0)$$

holds, instead of (7). The exact solution of this case of (1) is supplied by the real and imaginary parts of the function

$$x(t) = t \exp(i/t), \qquad (0 < t_0 \leqq t < \infty),$$

since (9) reduces to this function for the choice $f(t) = t^2$ (if i is replaced by $-i$ and a constant factor, introduced by the arbitrary integration limit $t_0 > 0$, is disregarded).

ON A ROUTING PROBLEM*

By RICHARD BELLMAN (*The RAND Corporation*)

Summary. Given a set of N cities, with every two linked by a road, and the times required to traverse these roads, we wish to determine the path from one given city to another given city which minimizes the travel time. The times are not directly proportional to the distances due to varying quality of roads and varying quantities of traffic.

The functional equation technique of dynamic programming, combined with approximation in policy space, yields an iterative algorithm which converges after at most $(N - 1)$ iterations.

1. Introduction. The problem we wish to treat is a combinatorial one involving the determination of an optimal route from one point to another. These problems are usually difficult when we allow a continuum, and when we admit only a discrete set of paths, as we shall do below, they are notoriously so.

The purpose of this paper is to show that the functional equation technique of dynamic programming, [1, 2], combined with the concept of approximation in policy space, yields a method of successive approximations which is readily accessible to either hand or machine computation for problems of realistic magnitude. The method is distinguished by the fact that it is a method of exhaustion, i.e. it converges after a finite number of iterations, bounded in advance.

2. Formulation. Consider a set of N cities, numbered in some arbitrary fashion from 1 to N, with every two linked by a direct road. The time required to travel from i to j is not directly proportional to the distance between i and j, due to road conditions and traffic. Given the matrix $T = (t_{ij})$, not necessarily symmetric, where t_{ij} is the time required to travel from i to j, we wish to trace a path between 1 and N which consumes minimum time.

Since there are only a finite number of paths available, the problem reduces to choosing the smallest from a finite set of numbers. This direct, or enumerative, approach is impossible to execute, however, for values of N of the order of magnitude of 20.

We shall construct a search technique which greatly reduces the time required to find minimal paths.

3. Functional equation approach. Let us now introduce a dynamic programming approach. Let

$$f_i = \text{the time required to travel from } i \text{ to } N, \text{ using an optimal policy}, \quad i = 1, 2, \cdots, N-1, \tag{3.1}$$

with $f_N = 0$.

Employing the principle of optimality, we see that the f_i satisfy the nonlinear system of equations

$$f_i = \operatorname*{Min}_{j \neq i} [t_{ij} + f_j], \qquad i = 1, 2, \cdots, N-1, \qquad f_N = 0. \tag{3.2}$$

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This system differs from the usual systems encountered in dynamic programming in that we do not have a ready computational scheme.

4. Uniqueness. Let us show that there exists at most one solution of the system in (3.2).

Assume that $\{f_i\}$ and $\{F_i\}$ are two solutions, with $f_N = F_N = 0$, and let k be an index for which $f_k - F_k$ achieves it maximum. Then

$$\begin{aligned} f_k &= \operatorname*{Min}_{j \neq k} [t_{kj} + f_j] \\ F_k &= \operatorname*{Min}_{j \neq k} [t_{kj} + F_j]. \end{aligned} \tag{4.1}$$

Let the minimum in the first equation be assumed for $j = r$, and the second for $j = s$. It is clear, since $t_{ij} > 0$ for all i, j, that $r \neq k$, $s \neq k$. Then we have the equalities and inequalities:

$$\begin{aligned} f_k &= t_{kr} + f_r \leq t_{ks} + f_s , \\ F_k &= t_{ks} + F_s \leq t_{kr} + F_r . \end{aligned} \tag{4.2}$$

These lead to

$$\begin{aligned} f_k - F_k &\leq f_s - F_s \\ &\geq f_r - F_r . \end{aligned} \tag{4.3}$$

Since k was an index where $f_k - F_k$ achieved its maximum, we must have

$$f_k - F_k = f_s - F_s , \tag{4.4}$$

which can only be true if in (2) we have

$$f_k = t_{ks} + f_s . \tag{4.5}$$

Now repeat this procedure for the pair $\{f_s , F_s\}$. It follows, from the foregoing argument, that there must be another pair $\{f_p , F_p\}$ with $f_p - F_p = f_s - F_s = f_k - F_k$. Furthermore, $p \neq s$, and $p \neq k$, since we have

$$f_k = t_{ks} + t_{sp} + f_p . \tag{4.6}$$

Proceeding in this way, we exhaust the set of points $i = 1, 2, \cdots, N - 1$, with the result that one of the terms in the continued equality above must be $f_N - F_N = 0$. Hence $f_i = F_i$ for $i = 1, 2, \cdots, N - 1$.

5. Approximation in policy space. Let us now turn to the problem of determining an algorithm for obtaining the solution of the system in (3.2). The basic method is that of successive approximations. We choose an initial sequence $\{f_i^{(0)}\}$, and then proceed iteratively, setting

$$\begin{aligned} f_i^{(k+1)} &= \operatorname*{Min}_{j \neq i} (t_{ij} + f_j^{(k)}], \qquad i = 1, 2, \cdots, N - 1, \\ f_N^{(k+1)} &= 0, \end{aligned} \tag{5.1}$$

for $k = 0, 1, 2, \cdots,$.

The choice of $\{f_i^{(0)}\}$ seems to require some care. Let us then invoke the concept of approximation in policy space in order to obtain a sequence which is monotone increasing.

Perhaps the simplest policy that we can employ is to proceed directly from i to N. Define

$$f_i^{(0)} = t_{iN}, \qquad i = 1, 2, \cdots, N. \tag{5.2}$$

It follows that $f_i^{(1)}$ as defined by

$$\begin{aligned} f_i^{(1)} &= \operatorname*{Min}_{j \neq i} [t_{ij} + f_j^{(0)}], \qquad i = 1, 2, \cdots, N-1, \\ f_N^{(1)} &= 0, \end{aligned} \tag{5.3}$$

satisfies the inequality

$$f_i^{(1)} \leq f_i^{(0)}, \qquad i = 1, 2, \cdots, N. \tag{5.4}$$

This inequality is immediate when we realize that $f_i^{(1)}$ represents the minimum time for a path with at most one stop. It follows then inductively that the sequences $\{f_i^{(k)}\}$ as defined by (5.1), with $f_i^{(0)}$ as in (5.2), satisfy the inequalities

$$f_i^{(k+1)} \leq f_i^{(k)}, \qquad i = 1, 2, \cdots, N, \qquad k = 0, 1, 2, \cdots \tag{5.5}$$

It is important to note that there are many other policies we could employ to obtain monotone convergence.*

It follows that

$$\lim_{k \to \infty} f_i^{(k)} = f_i, \qquad i = 1, 2, \cdots, N, \tag{5.6}$$

furnishing a solution to (3.2).

It is clear from the physical interpretation of this iterative scheme that at most $(N - 1)$ iterations are required for the sequence to converge to the solution.

6. Computational aspects. It is easily seen that the iterative scheme discussed above is a feasible method for either hand or machine computation for values of N of the order of magnitude of 50 or 100.†

For each i, we require only the column (t_{ij}), $j = 1, 2, \cdots, N$ of the matrix T. Hence, the memory requirement for a digital computer is small.

7. Monotone increasing convergence. Turning back to (3.2), let us consider the sequence of approximations defined by

$$\begin{aligned} f_i^{(0)} &= \operatorname*{Min}_{j \neq i} t_{ij}, \qquad i = 1, 2, \cdots, N = 1, \\ f_N^{(0)} &= 0, \\ f_i^{(k+1)} &= \operatorname*{Min}_{j \neq i} [t_{ij} + f_j^{(k)}], \qquad i = 1, 2, \cdots, N-1, \\ f_N^{(k+1)} &= 0. \end{aligned} \tag{7.1}$$

It is clear that $f_i^{(k+1)} \geq f_i^{(k)}$. It is not, however, obvious that this method yields a uniformly bounded sequence. To establish this, let us show, inductively, that

$$f_i^{(k)} \leq f_i, \qquad i = 1, 2, \cdots, N, \qquad k = 0, 1, 2, \cdots, \tag{7.2}$$

where $\{f_i\}$ is the solution of (3.2).

*I owe this choice of an initial policy to F. Haight.

†*Added in proof* (*December* 1957): After this paper was written, the author was informed by Max Woodbury and George Dantzig that the particular iterative scheme discussed in Sec. 5 had been obtained by them from first principles.

The inequality is certainly true for $k = 0$. Hence, assuming that it holds for k, we have

$$f_i^{(k+1)} = \underset{i \neq i}{\text{Min}} [t_{ii} + f_i^{(k)}] \leq \underset{i \neq i}{\text{Min}} [t_{ii} + f_i] \leq f_i . \tag{7.3}$$

It follows that the sequence $\{f_i^{(k)}\}$ converges to $\{f_i\}$ as $k \to \infty$, furnishing the desired monotone convergence. Once again, only a finite number of iterations will be required. It is to be expected that the first method will converge more rapidly.

References

1. R. Bellman, *Dynamic Programming*, Princeton University Press, 1957
2. R. Bellman, *The Theory of Dynamic Programming*, Bull. Amer. Math. Soc., **60**, 503-515 (1954)

ON THE DAMPED OSCILLATIONS EQUATION WITH VARIABLE COEFFICIENTS*

By E. V. LAITONE (*University of California, Berkeley*)

A useful upper bound can be given for the oscillatory solutions of the second-order linear differential equation with continuous differentiable coefficients

$$u''(t) + p(t)u'(t) + q(t)u(t) = 0 \tag{1}$$

which commonly occurs in vibration studies and airplane or missile dynamics.

The solutions of Eq. (1) are oscillatory whenever

$$\phi(t) = (q - p^2/4 - p'/2) \geqq m^2 > 0 \tag{2}$$

see Bellman [1] or Kamke [2], and for this case it will be shown that as long as ϕ is monotonic the solutions are bounded in the following manner:

$$| u(t) | \leqq \frac{1}{m} \{\phi(0)u(0)^2 + [u'(0) + p(0)u(0)/2]^2\}^{1/2} \exp\left(-\int_0^t p/2 \, dt\right). \tag{3}$$

This result can be derived by introducing the unique transformation, see Bellman [1] or Kamke [2]

$$u = v \exp\left(-\int_0^t p/2 \, dt\right) \tag{4}$$

which preserves the same zeros in the oscillatory solutions. Then Eq. (1) is transformed to

$$v''(t) + \phi(t)v(t) = 0, \tag{5}$$

*Received February 5, 1957.

where ϕ is defined by Eq. (2). The inequality $\phi \geqq m^2 > 0$ insures that all the solutions are oscillatory, see Bellman [1] or Kamke [2], also it is known that if in addition ϕ is monotonic, then the amplitudes of v also vary monotonically, increasing when ϕ is decreasing and vice-versa.

Then the function

$$F(t) = \phi v^2 + v'^2 \tag{6}$$

is introduced, see Kamke [2], in order to develop the inequality

$$\frac{F'(t)}{F(t)} = \frac{2\phi vv' + \phi' v^2 + 2v'v''}{\phi v^2 + v'^2} = \frac{\phi' v^2}{\phi v^2 + v'^2} \leqq \frac{\phi'(t)}{\phi(t)} \tag{7}$$

which will be used to derive Eq. (3).

Case I, ϕ monotonically decreasing.

Since $\phi' < 0$, therefore $F' = \phi' v^2 \leqq 0$ so that

$$F(0) \geqq F(t) \geqq \phi(t) v(t)^2 \geqq m^2 v(t)^2. \tag{8}$$

Consequently

$$| v(t) | \leqq \frac{1}{m} F(0)^{1/2} = \frac{1}{m} [\phi(0) v(0)^2 + v'(0)^2]^{1/2} \tag{9}$$

which transforms to Eq. (3) upon introducing Eq. (4).

Case II, ϕ monotonically increasing.

Since $\phi' > 0$, therefore $\phi(t) > \phi(0) \geqq m^2$ and $F' = \phi' v^2 \geqq 0$ so that

$$F(t) \geqq F(0) \geqq \phi(0) v(0)^2 \geqq m^2 v(0)^2. \tag{10}$$

Consequently Eq. (7) may be integrated to

$$\ln \frac{\phi(t)}{\phi(0)} \geqq \ln \frac{F(t)}{F(0)} \geqq \ln \frac{\phi(t) v(t)^2}{\phi(0) v(0)^2 + v'(0)^2}. \tag{11}$$

This shows that if $v^2 < 1$ and

$$\phi(0) > F(0) > 0 \tag{12}$$

then Eq. (11) again yields Eq. (9), and similarly, Eq. (3). Now $F > 0$ for $\phi > 0$ since v and v' cannot be simultaneously zero, however $\phi > F$ only for sufficiently small initial displacements from equilibrium so that $v^2 < 1$ and $v'^2 < (1 - v^2)\, \phi \leqq \phi$.

Consequently the bound provided for Eq. (1) by Eq. (3) is always valid when $\phi(\geqq m^2 > 0)$ is monotonically decreasing, but is only valid for sufficiently small initial values of u and u' so that Eq. (12) is satisfied when ϕ is monotonically increasing. This latter requirement is satisfied whenever

$$\phi(0) > \{\phi(0) u(0)^2 + [u'(0)^2 + p(0) u(0)/2]^2\} \tag{13}$$

in which case Eq. (3) could be replaced by

$$| u(t) | \leqq \frac{1}{m} \phi(0)^{1/2} \exp\left(-\int_0^t p/2\, dt\right) \tag{14}$$

or in the usual situation wherein $\phi(0) = m^2$ when ϕ is monotonically increasing, by

$$| u(t) | \leqq \exp\left(-\int_0^t p/2\, dt\right). \tag{15}$$

The limiting behavior of Eq. (5) as $t \to \infty$, see Bellman [1] or Kamke [2], shows that in both cases the solutions of Eq. (1) can be written for $t \to \infty$ as

$$u \to C_1 \sin(at + C_2) \exp\left(-\int_0^t p/2\, dt\right), \tag{16}$$

whenever $\phi(t) \to a^2 (\geqq m^2 > 0)$ as $t \to \infty$ and

$$\int_0^\infty \left| \frac{\phi(t)}{a^2} - 1 \right| dt < \infty.$$

Application to dynamic stability analysis of a missile. As a typical application of the upper bound provided by Eq. (3) let us consider the case of a missile having a nearly vertical trajectory with no control forces or thrust misalignment. In this case the equations presented in [3], which define the variation of the small incremental angle of attack α, during constant acceleration, may be written as

$$\alpha''(t) + p(t)\alpha'(t) + q(t)\alpha(t) = 0. \tag{17}$$

Therefore as long as p and q are such that the ϕ defined by Eq. (2) is monotonic, the upper bound of α can be determined. This is generally true for constant atmospheric density since the forward velocity U is continually increasing while the rate of acceleration is generally small. However if the magnitude of the acceleration is sufficiently large, or the height above the earth's surface is rapidly varying, then it is necessary to transform the independent variable from the time to the distance along the trajectory, as in [3], before Eq. (3) can be applied.

A particularly interesting case occurs for the vertical entry of a missile into the earth's atmosphere at hypersonic speeds. Then the ratio of the acceleration to the velocity squared is nearly constant so that in terms of χ, the non-dimensional distance along the trajectory normal to the earth's surface, the variation of the small incremental angle of attack may be approximated to the first order by

$$\alpha''(\chi) + b\rho(\chi)\alpha'(\chi) + c\rho(\chi)\alpha(\chi) = 0, \tag{18}$$

where b and c are constants and ρ is the mass density of the air.

Consequently, Eq. (2) becomes

$$\phi(\chi) = [c\rho - (b\rho/2)^2 - b\rho'/2]$$

and it is easily shown that $\phi(\chi)$ is always bounded and monotonically increasing for any physically possible variation in air density as long as $\phi(0) > 0$ for the constant mass missile at hypersonic speeds.

For example, the usual exponential approximation to the density variation may be written in terms of ρ_∞, the sea-level mass density of the air, as

$$\rho(\chi) = \rho_\infty\{1 - \exp[-\beta(\chi + \chi_0)]\}$$

so that ϕ is monotonically increasing and bounded by

$$0 < \phi(0) \leqq \phi(\chi) < [c\rho_\infty - (b\rho_\infty/2)^2 - b\beta\rho_\infty/2]$$

since

$$\rho_\infty \ll 1 \text{ lb. sec.}^2 \text{ ft.}^{-4}$$

$$0 < \beta \ll 1 \text{ ft.}^{-1},$$

for any physically possible atmosphere.

Therefore $\alpha(\chi)$ is oscillatory as long as $\phi(0) > 0$, and the upper bound is defined by Eq. (3) as long as

$$\phi(0) > \{\phi(0)\alpha(0)^2 + [\alpha'(0)^2 + b\rho(0)\alpha(0)/2]^2\},$$

that is, provided Eq. (13) is satisfied.

References

1. R. Bellman, *Stability theory of differential equations*, McGraw-Hill, N. Y., 1953, Chap. 6
2. E. Kamke, *Differentialgleichungen*, vol. I, *Gewöhnliche Differentialgleichungen*, 3d ed., Chelsea Publishing Co., 1948, pp. 119-138
3. T. W. Oswald, *Dynamic behaviour during accelerated flight with particular application to missile launching*, J. Aeronaut. Sci. **23**, 781-791 (1956)

BOOK REVIEWS

(*Continued from p. 72*)

The mathematics of diffusion. By J. Crank. Oxford University Press, New York, 1956. vi + 347 pp. $8.00.

This is the first book on the diffusion-heat conduction equation to appear since the publication of the well known work by Carslaw and Jaeger. Almost all the situations treated involve the transport of matter rather than of heat, and many show a number of features, such as concentration dependence of the diffusion coefficient, which are normally unimportant in the corresponding heat conduction problems. The diffusion processes are for the most part one dimensional, although quite a few cases with spherical and cylindrical symmetry are presented.

The first six chapters constitute a necessary review of fairly standard diffusion problems, but from this point on most of the material has never been presented in book form before; indeed, much of it is from the author's own publications. The chief topics discussed are moving boundary problems, diffusion with simultaneous chemical reaction, concentration-dependent diffusion coefficients, and numerical methods. In addition there is a chapter dealing with the different kinds of diffusion coefficients measured by various experimental procedures, plus one on the somewhat specialized problem of coupled diffusion and heat conduction in sheets of wool fiber.

One of the best features of the book is the inclusion of a very large number of plots of concentration-distance and sorption-time relationships; these greatly aid the reader's intuition, and many of them should be useful in actual computations. Also mentioned should be the appendix, which contains tables of a variety of useful quantities, such as integrals of the error function, as well as sample computation sheets for some of the numerical methods presented in the text.

If there is anything to be aid by way of criticism, it is not about the material which has been included, but rather about that which has been left out. Thus, coupling between the diffusion of different species in multicomponent systems is not mentioned, and no problems involving convection or external forces are treated. Perhaps the inclusion of these and other topics would have made the book excessively long. In any case, it is a valuable contribution as it stands, and no one engaged in or contemplating diffusion measurements should be unfamiliar with its contents.

STEPHEN PRAGER

Introduction to finite mathematics. By John G. Kemeny, J. Laurie Snell, and Gerald L. Thompson. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1957. xi + 372 pp. $5.00.

Here is a book which can safely be described as unique among college mathematics texts.

In recent years there has been an increasing demand for "terminal" courses in mathematics, running for one or at most two semesters and intended to provide appropriate material for students who may not intend to go further in mathematics. The usual introductory analytic geometry and calculus course is from some points of view unsatisfactory for this purpose, firstly, because one or two terms is hardly enough time to get started in the subject, and secondly, the material lacks motivation without a considerable amount of application to the physical sciences. "Introduction to Finite Mathematics" overcomes both of these difficulties, first, by taking for its mathematical subject matter not analysis but "finite mathematics," (actually certain branches of algebra), and second, by choosing motivating material from the social rather than the physical sciences, thus dealing with matters which are close to the students' every day experience.

In the authors' own words the book's purpose is "to introduce a student to some concepts in modern mathematics early in his college career. While primarily a mathematics course, it was to include applications to the biological and social sciences and thus provide a point of view, other than that given by physics, concerning the possible uses of mathematics. Our aim was to choose topics which were initially close to the student's experience, which are important in modern day mathematics, and which have interesting and important applications."

As to the somewhat mystifying title of the book, the authors explain "our purpose . . . was to develop

several topics from a central point of view. In order to accomplish this on an elementary level, we restricted ourselves to consideration of *finite* problems, that is, problems which do not involve infinite sets, limiting processes, continuity, etc."

The program of the book in very brief outline, is the following: The first chapter "Compound Sentences," deals with the elementary propositional calculus, the central idea being that of truth tables. Also in this chapter the notion of the "logical possibilities" associated with a statement is introduced. This concept which appears to be a device invented by the authors (we return to this later) provides a link between the first and second chapter entitled "Sets and Subsets." Its main concern is the Boolean Algebra of sets and its relation to the propositional calculus of the previous chapter. The set notion leads naturally to Chapter III, "Partitions and Counting," which is concerned with elementary combinatorics, permutations, etc. The material of all the preceding chapters is brought together in Chapter IV, "Probability Theory," which, however, is quite a departure from the usual treatments of this subject in first year texts as indicated by the presence of sections entitled "Finite Stochastic Processes," "The Law of Large Numbers," "Markov Chains." This last section leads to Chapter V on "Vectors and Matrices" which, after introducing the standard terms and operations, applies them to the theory of Markov chains. These five chapters comprise the "basic core" of the book and represent material which could be conveniently covered in one college semester. The book concludes with two "optional" chapters, (so indicated by an asterisk). The first entitled "Linear Programming and the Theory of Games," contains an elementary treatment of some aspects of these two subjects. The final chapter "Applications to Behavioral Science Problems" treats five problems, "selected for their interest both to mathematicians and to behavioral scientists. One topic was chosen from each of five sciences; sociology, genetics, psychology, anthropology, and economics."

This book represents such a radical departure from traditional college texts that it would take considerable space to give it a really adequate review. Perhaps some notion of its unusual flavor can be conveyed by listing a few of the many (over 1000) varied and excellent problems which appear at the ends of each section.

In a starred (optional) section of Chapter I on "Statements Having Given Truth Tables" the student is lead in a series of exercises to the proof that every logical connective can be expressed in terms of "neither . . . nor." In a section on "Valid Arguments" the authors develop the theory of the syllogism so the student has a systematic procedure for checking the validity of arguments like "Father praises me only if I can be proud of myself. Either I do well in sports or I cannot be proud of myself. If I study hard then I cannot do well in sports. Therefore, if Father praises me then I do not study hard."

The final section (starred) of this chapter is on "Application to Switching Circuits". Problem: "A committee has five members. It takes a majority vote to carry a measure except that the chairman has a veto. Design a circuit for the committee, so that each member votes for a measure by pressing a button, and a light goes on if and only if the measure is carried."

In the chapter on sets the following problem is typical. "In a survey of 100 students the number studying various languages were found to be: Spanish, 28; German, 30; French, 42; German and French, 5; all three languages, 3. (a) How many students were studying no language? (b) How many students had French as their only language?" etc. In the chapter on partitions we find, "On a transcontinental airliner there are 9 boys, 5 American children, 9 men, 7 foreign boys, 14 Americans, 6 American males and 7 foreign females. What is the number of people on the plane?"

As a final example we quote one of the problems on Markov chains. "A professor tries not to be late for class too often. If he is late one day, he is 90% sure to be on time the next. If he is on time, then the next day there is a 30% chance of his being late. In the long run how often is he late?".

The foregoing examples show the way in which each new mathematical concept is illustrated by a non-trivial application. Even such seemingly mechanical matters as the rules for matrix multiplication are illustrated by realistic examples in which multiplying matrices is precisely the natural thing to do. Indeed, of the text's many virtues, perhaps the outstanding one is the remarkable balance which the authors have preserved between abstract theory and "common sense" applications. This is "applied mathematics" in the best sense.

A second notable feature of the book is its emphasis on pedagogical devices of a graphical nature. Simple problems in set theory are solved by the use of Venn diagrams. Another device which occurs in the first chapter and is used especially effectively in the probability chapter is that of "three diagrams" to represent either partitions, or "logical possibilities" at various stages of some process. This use of "picture writing" is thoroughly exploited to the great advantage of the student.

Having pointed out some of the books strong points, we now turn to what, in the reviewer's opinion, is a very serious flaw. The authors have chosen as their central notion the concept of a "statement." Almost all other notions in the book are derived from or closely related to this one. In the chapter on probability the fundamental notion is not the usual idea of the probability of an event but the "probability of a statement." It is therefore somewhat of a shock to find that this notion is used in a completely inconsistent manner. Specifically, the book opens with the following sentences, "A *statement* is a verbal or written assertion. In the English language such assertions are made by means of declarative sentences." But on page 21 we find, "Hence, also, the *statement* [the italics are the reviewer's] "Draw three white balls" is logically false." (Problem for the authors: what is a declarative sentence?). Next, on page 2 we find "The fundamental property of any statement is that it is either true or false (and that it cannot be both true and false)," but on page 19 we read "Normally, for a given statement there will be many cases in which it is true and many in which it is false." Which sentence is the student to believe? Worse, what is the instructor to tell him? This example does not represent a momentary lapse on the authors' part, but a deliberate ambiguity which is carried through the whole book.

There is no need to dwell on the morass of inconsistency caused by the authors' refusal to distinguish between sentences like "$x^2 = 4$" and "p implies q" which are *not* statements (they are "sentential functions") and are true in some cases, false in others, and honest to goodness statements like "Socrates is a man" and "$2 + 2 = 5$" which are either true or false,—no cases about it. The result of the ambiguity, however, is to put the instructor in an unhappy dilemma. In the authors' phraseology, he will be faced with the following "logical (or pedagocial) possibilities" (we refrain from drawing the tree diagram) he must either play along with the authors, hoping the students won't detect the deception, or he must rewrite a substantial portion of the exposition himself. It is unfortunate that this quite unnecessary mix-up had to occur in what is in many ways a superlative text. Perhaps these matters can be remedied in an eventual second edition.

The last two chapters of the book should really be considered as a separate unit since the level of difficulty, especially in the final chapter, is much higher than in the first five. The reviewer found the exposition here rather muddy in spots, especially the section on Este's learning model.

The examples treated were, however, uniformly interesting and, I suspect, entirely unfamiliar to nine out of ten teachers of college mathematics.

Finally, this book should be welcomed eagerly by college teachers in departments other than mathematics, especially behavioral scientists, as it provides precisely the sort of elementary mathematical training which will be useful to students whose main interest lies in this field.

DAVID GALE

Physics and mathematics. By R. A. Charpie, J. Horowitz, D. J. Hughes, and D. J. Littler. McGraw-Hill Book Co., Inc., New York, and Pergamon Press Ltd., London, 1956. x + 398 pp. $12.00.

This book is labeled Volume One of Series One (of eight series) arising mainly as the result of the review articles presented at the Geneva Conference. This volume contains articles on neutron physics (especially cross sections) mainly in connection with the physics of fission. There are eleven chapters with titles and authors as follows (1) Summary of Data on Cross Sections and Neutron Yields of U^{233}, U^{235}, and Pu^{239}—Harvey and Sanders; (2) Resonance Structure of U^{233}, U^{235}, and Pu^{239}—Egelstaff and Hughes; (3) Theoretical Analysis of Neutron Resonances in Fissile Materials—Bethe; (4) Techniques for Measuring Elastic and Non-Elastic Neutron Cross Sections—Cranberg, Day, Rosen, Taschek and Walt; (5) The Cross Section of the Fission Product Poison x_e^{135} as a Function of Energy—Bernstein and Smith; (6) Resonance Capture Integrals—Macklin and Pomerance; (7) Delayed Neutrons—Keepin; (8) Homogeneous Critical Assemblies—Callihan; (9) The Physics of Fast Reactors—Codd, Sheperd and Tait; (10) Heterogeneous Methods for Calculating Reactors—Feinberg; (11) Highly Enriched Intermediate and Thermal Assemblies—Hurwitz and Ehrlich.

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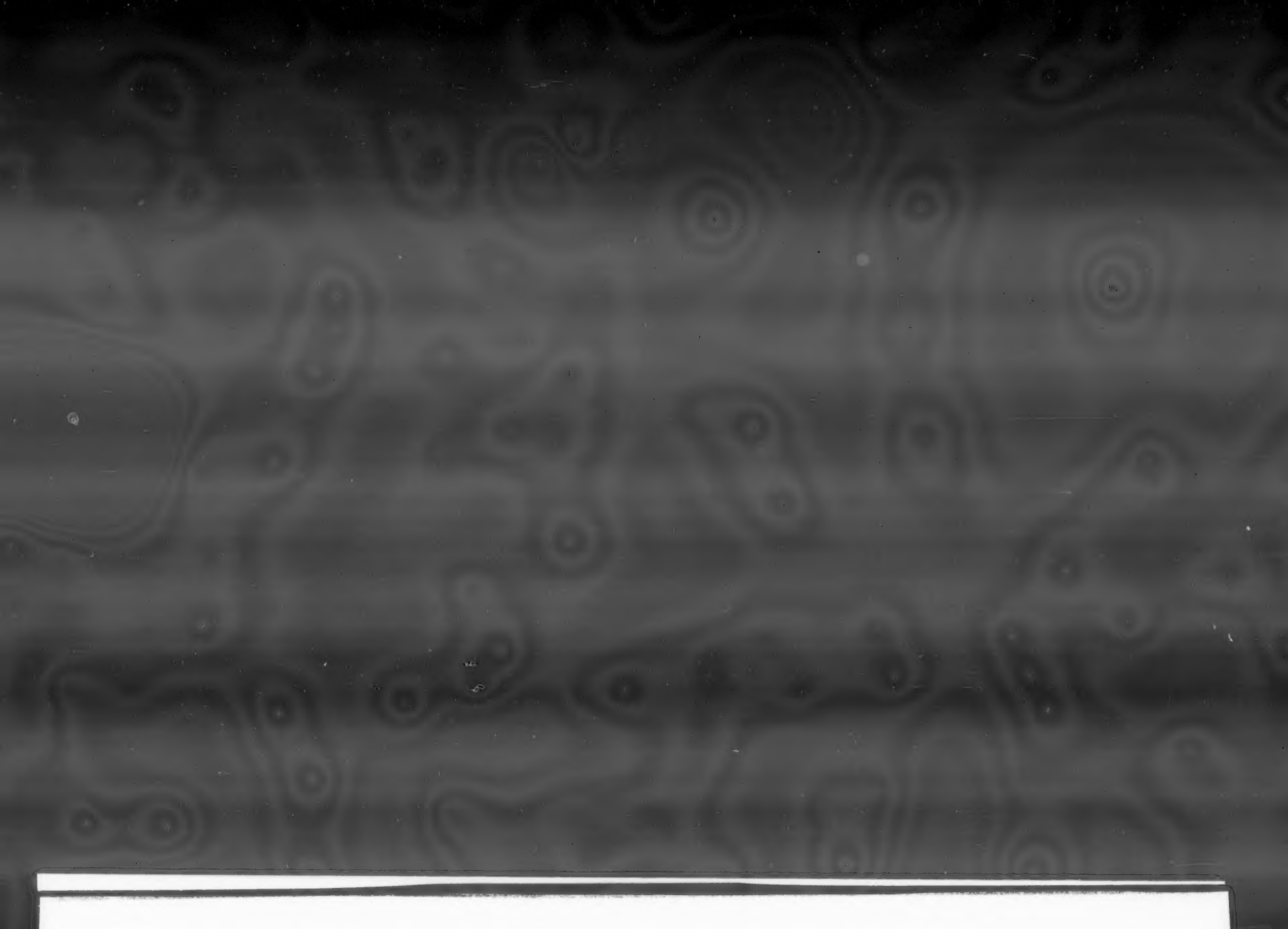

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